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AN AXIOMATIC APPROACH TO LOCATION ON NETWORKS*†

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The problem under consideration is that of locating a facility on a tree-network, given data specifying the locations of the users on the network. The approach taken is to formulate axioms that require consistent response of the solution to variations in the users' location data. It is shown that three independent axioms determine together a unique solution, located at the point that minimizes the sum of the squares of the distances to the users.

1. Introduction. Several users, with specified locations along the roads of a given transportation network, are interested in a public facility (say, a library), to be located somewhere along these roads. Each user would like the facility to be located as near as possible to his/her own location. The problem is what location for the facility is "best", in some sense.

This problem has been dealt with extensively; the book by Handler and Mirchandani [5] reflects the literature on the problem. The classical approach is to set up an objective function and then solve the respective optimization problem. More specifically, two solution concepts dominate the literature:

(i) *Minisum*—a location is best if it minimizes the sum of the distances of the users to the facility.

(ii) *Minimax*—a location is best if it minimizes the maximum distance of the users to the facility.

The choice of a particular objective function is rather ad hoc. It is usually justified by referring to additional attributes of the problem, not part of the formal model. Here we propose an axiomatic approach: a particular solution concept is justified in terms of its internal consistency as displayed in the way it treats different instances of the problem.

The axiomatic approach has been used successfully in the theory of cooperative games. Perhaps the best known solution concepts obtained with this approach are the Nash [8] solution for the bargaining problem and the Shapley [11] value for side-payment games. The main purpose of the present paper is to show that this approach can be applied to the network location problem.

In §2 we describe the model formally and state three axioms that we regard as sensible consistency requirements. In §3 we state and prove our main result: there

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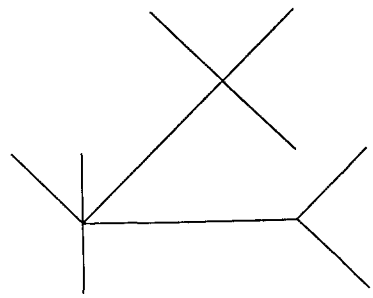

FIGURE 1. An Infrastructure with the Tree Property.

exists a unique solution concept that satisfies the three axioms. We investigate the logical independence of the axioms in §4. We further discuss our approach and the solution it yields and point out possible continuations of this work in §5.

We note that our attention is restricted to tree-networks, which is a significant limitation (though not uncommon in the literature). Also, the particular axioms may come under criticism, and other axioms may be suggested, leading possibly to other solutions. Yet, we insist on the merits of our methodology, namely that a solution concept should be based on an axiomatic foundation.

2. The model and the axioms. The *infrastructure* is described by a set of the form $T = \bigcup_{k=1}^{K}\gamma_k$, where γ_k are rectifiable curves in the plane E^2, with the *tree property*: for every two points $x, y \in T$ with $x \neq y$ there exists a unique *route linking x and y* (i.e., an inclusion minimal subset of T that contains x and y and is connected). An example of such a set T is depicted in Figure 1.

Here are a few remarks about this formalization:

1. Some authors assign a distinguished set of "vertices", but this is arbitrary since any point can be declared a vertex without modifying the real infrastructure. By not making the set of vertices part of the formal description, we are implicitly assuming that the solution is invariant with respect to such declarations.

2. The planarity of the infrastructure plays no essential role in our work, and is assumed only for the sake of illustration.

3. The tree property amounts to connectedness and absence of "cycles". By making this assumption we avoid the difficulties caused by cycles, but our problem is still far from being trivial. We discuss the question of extending our work to infrastructures without the tree property in §5.

For $x, y \in T$ with $x \neq y$, let $R\{x, y\}$ denote the unique route linking x and y. Its length, defined as the limit of the lengths of approximating polygonal curves, is the *distance from x to y*, denoted $d(x, y)$; for $x = y$ we let $d(x, y) = 0$. Then (T, d) is a connected and compact metric space, which generates the same topology as the one inherited from the plane. For $x \in T$, each connected component of $T \setminus \{x\}$ is a *direction from x*.

The demand side is described by a set P, the *population*, whose elements are indexed by $1, \ldots, p$; a vector $x^P = (x^1, \ldots, x^p) \in T^P$ specifies the locations of the individuals. In our treatment we shall view the infrastructure T and the population P as fixed, and let the vector x^P vary over T^P. As x^P varies, we are considering a whole family of location problems which we attempt to solve in a consistent way.

A *solution* is a function l: $T^P \to T$. The interpretation is that given any individual locations $x^P \in T^P$, the best location for the facility according to the solution is $l(x^P)$.

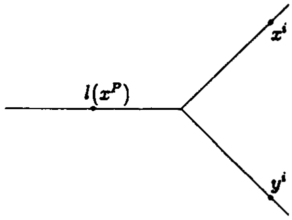

FIGURE 2. The Invariance Axiom.

We proceed to formulate three axioms that we regard as sensible requirements for a solution:

Unanimity ($\mathscr{U}$). $\forall x \in T$, $l(x, \ldots, x) = x$.

Lipschitz ($\mathscr{L}$). $\forall i \in P$, $\forall x^P, y^P \in T^P$, if $\forall j \neq i$, $x^j = y^j$ then

$$d(l(x^P), l(y^P)) \leqslant \frac{1}{p} d(x^i, y^i).$$

Invariance ($\mathscr{I}$). $\forall i \in P$, $\forall x^P, y^P \in T^P$, if $\forall j \neq i$, $x^j = y^j$ and [y^i is at the same distance and in the same direction form $l(x^P)$ as x^i] then $l(y^P) = l(x^P)$.

The first axiom hardly needs any comments. The second one, a Lipschitz condition, can be understood as a strong type of continuity requirement: the solution should not be too sensitive to small changes in the data (perhaps due to errors of measurement). To illustrate the motivation for such a requirement, consider the case where T is an interval with endpoints x and y, and the population splits into two parts: 1000 individuals are located at x and 999 are located at y. Assume further that the minisum solution (see the Introduction) is employed. Then the solution point is x. But if one individual moves from x to y, then so does the solution point. It seems strange that a relatively insignificant change in the data should cause a drastic change in the solution point. The axiom captures the idea that we expect a smooth behavior of the solution. The particular Lipschitz constant, $1/p$, stands for the following idea: no individual should be able to influence the outcome by more than his/her proportional share in the population.

The third axiom is illustrated in Figure 2. As i moves from x^i to y^i, which is equidistant from $l(x^P)$, we should not expect the solution point to change (it is understood that the locations of the other individuals, not shown in the figure, stay put). The rationale behind the axiom is that a movement that preserves the distance and the direction from the solution point does not affect in any way the mover's claim with respect to the solution point. We note that the preservation of distance only is not sufficient in this argument. Indeed, when two individuals are located at different points, we expect the solution point to be half way between them; but when one of them moves to the other's location, the solution point should move there too.

3. The theorem. Let T be an infrastructure with the tree property, and let $P = \{1, \ldots, p\}$ be a population.

THEOREM. *There exists a unique solution that satisfies the axioms* ($\mathscr{U}$), ($\mathscr{L}$) *and* ($\mathscr{I}$). *This solution is the function* $\bar{l}$: $T^P \to T$ *defined by*:

$$\bar{l}(x^P) = \underset{x \in T}{\arg\min} \sum_{i=1}^{p} \left[d(x^i, x)\right]^2.$$

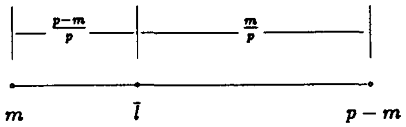

FIGURE 3. The Performance of $\bar{l}$ in a Special Case.

Before the proof, we illustrate the solution $\bar{l}$ for situations where the population splits into two subpopulations of sizes m and $p - m$, located at the endpoints of the unit interval. In this case, the solution point splits the interval proportionally; see Figure 3. By letting m vary through $m = 0, \ldots, p$, one can see that $(\mathscr{U})$ and $(\mathscr{L})$ completely determine the solution for these situations. This is a simple form of the argument for uniqueness of the solution in the general case, which appears later in the proof.

In the proof we shall establish that $\bar{l}$ is well-defined, that it satisfies the axioms and that any solution which satisfies the axioms must coincide with it. We shall proceed by steps, and start with some notation and terminology.

For $x \in T$, let $\mathscr{D}(x)$ denote the set of all directions from x. For $x \in T$, $D \in \mathscr{D}(x)$ and $x^P \in T^P$, let:

$$P_D(x^P) = \{i \mid x^i \in D\}.$$

$$\Sigma_D(x^P, x) = \sum_{i \in P_D(x^P)} d(x^i, x),$$

$$\Sigma_{\setminus D}(x^P, x) = \sum_{i \notin P_D(x^P)} d(x^i, x).$$

By convention, sums over the empty set are 0. When there is only one x^P under consideration, we shall use the shorter notations P_D, $\Sigma_D(x)$ and $\Sigma_{\setminus D}(x)$.

Given $x^P \in T^P$, we shall say that $x \in T$ is *balancing* if

$$\Sigma_D(x^P, x) \leqslant \Sigma_{\setminus D}(x^P, x), \quad \forall D \in \mathscr{D}(x);$$

we shall say that $x \in T$ is *minimizing* if

$$\sum_{i=1}^{p} \left[d(x^i, x)\right]^2 \leqslant \sum_{i=1}^{p} \left[d(x^i, y)\right]^2, \quad \forall y \in T.$$

Step 1. A minimizing point is also balancing.

To see this, assume that for some $D \in \mathscr{D}(x)$ we have $\Sigma_D(x) > \Sigma_{\setminus D}(x)$. Let y be a point such that $y \in R\{x, x^i\}$ for every $i \in P_D$. Denoting $d(x, y) = \delta$ we have $d(x^i, y) = d(x^i, x) - \delta$ for $i \in P_D$ and $d(x^i, y) = d(x^i, x) + \delta$ for $i \notin P_D$. Hence

$$\sum_{i=1}^{p} \left[d(x^i, x)\right]^2 - \sum_{i=1}^{p} \left[d(x^i, y)\right]^2$$

$$= \sum_{i \in P_D} \left[2d(x^i, x)\delta - \delta^2\right] - \sum_{i \notin P_D} \left[2d(x^i, x)\delta + \delta^2\right]$$

$$= 2\delta\left[\Sigma_D(x) - \Sigma_{\setminus D}(x)\right] - p\delta^2.$$

The last expression can be made positive if y is chosen so that δ is positive but sufficiently small. Hence x is not minimizing.

Step 2. There is at most one balancing point.

Indeed, suppose that both x and y are balancing and $d(x, y) = \delta > 0$. Let D and D' be defined by $y \in D \in \mathscr{D}(x)$ and $x \in D' \in \mathscr{D}(y)$. We have

$$\Sigma_D(x) \geqslant \Sigma_{\setminus D'}(y) + (p - |P_{D'}|)\delta, \quad \Sigma_{D'}(y) \geqslant \Sigma_{\setminus D}(x) + (p - |P_D|)\delta.$$

By assumption $\Sigma_D(x) \leqslant \Sigma_{\setminus D}(x)$ and $\Sigma_{D'}(y) \leqslant \Sigma_{\setminus D'}(y)$, which is compatible with the above inequalities only if the terms involving δ vanish. Since $\delta > 0$, this requires that $x^i \in D \cap D'$ for all $i \in P$. But then $\Sigma_D(x) > 0$ and $\Sigma_{\setminus D}(x) = 0$, contrary to x being balancing.

Step 3. $\bar{l}$ is well-defined; $\bar{l}(x^P)$ is the unique minimizing point as well as the unique balancing point.

The existence of a minimizing point follows from the compactness of (T, d) and the continuity of $\Sigma_{i=1}^{p}[d(x^i, \cdot)]^2$. The rest follows from Steps 1 and 2.

Step 4. $\bar{l}$ satisfies $(\mathscr{U})$.

This is obvious.

Step 5. $\bar{l}$ satisfies $(\mathscr{L})$.

Suppose the axiom is violated. Then we can find $i \in P$ and $x^P, y^P \in T^P$ such that $\forall j \neq i \ x^j = y^j$ but $d(\bar{x}, \bar{y}) > (1/p)d(x^i, y^i)$, where $\bar{x} = \bar{l}(x^P)$ and $\bar{y} = \bar{l}(y^P)$. Let $\delta = d(\bar{x}, \bar{y})$ and let D and D' be defined by $\bar{y} \in D \in \mathscr{D}(\bar{x})$ and $\bar{x} \in D' \in \mathscr{D}(\bar{y})$. We have

$$\Sigma_D(x^P, \bar{x}) = \Sigma_{\setminus D'}(x^P, \bar{y}) + \left(p - \left|P_{D'}(x^P)\right|\right)\delta + \sum_{j \in P_D(x^P) \cap P_{D'}(x^P)} d(x^j, \bar{x}),$$

$$\Sigma_{D'}(x^P, \bar{y}) = \Sigma_{\setminus D}(x^P, \bar{x}) + \left(p - \left|P_D(x^P)\right|\right)\delta + \sum_{j \in P_D(x^P) \cap P_{D'}(x^P)} d(x^j, \bar{y}).$$

Using the fact that $\bar{x}$ is balancing for x^P we get

$$\begin{aligned}
\Sigma_{D'}(x^P, \bar{y}) &\geqslant \Sigma_D(x^P, \bar{x}) + \left(p - \left|P_D(x^P)\right|\right)\delta + \sum_{j \in P_D(x^P) \cap P_{D'}(x^P)} d(x^j, \bar{y}) \\
&= \Sigma_{\setminus D'}(x^P, \bar{y}) + \left(2p - \left|P_D(x^P)\right| - \left|P_{D'}(x^P)\right|\right)\delta \\
&\quad + \sum_{j \in P_D(x^P) \cap P_{D'}(x^P)} \left[d(x^j, \bar{x}) + d(x^j, \bar{y})\right] \\
&\geqslant \Sigma_{\setminus D'}(x^P, \bar{y}) + \left(2p - \left|P_D(x^P)\right| - \left|P_{D'}(x^P)\right| + \left|P_D(x^P) \cap P_{D'}(x^P)\right|\right)\delta \\
&= \Sigma_{\setminus D'}(x^P, \bar{y}) + p\delta.
\end{aligned}$$

As we turn our attention from x^P to y^P, the difference $\Sigma_{D'}(\bar{y}) - \Sigma_{\setminus D'}(\bar{y})$ changes by no more than $d(x^i, y^i)$. Therefore

$$\begin{aligned}
\Sigma_{D'}(y^P, \bar{y}) - \Sigma_{\setminus D'}(y^P, \bar{y}) &\geqslant \Sigma_{D'}(x^P, \bar{y}) - \Sigma_{\setminus D'}(x^P, \bar{y}) - d(x^i, y^i) \\
&\geqslant p\delta - d(x^i, y^i) > 0,
\end{aligned}$$

which contradicts the fact that $\bar{y}$ is balancing for y^P.

Step 6. $\bar{l}$ satisfies ($\mathscr{I}$).

This is true because the change considered in the axiom preserves the property of being balancing.

Step 7. No solution other than $\bar{l}$ satisfies ($\mathscr{U}$), ($\mathscr{L}$) and ($\mathscr{I}$).

Suppose that l is a solution which satisfies the axioms, but differs from $\bar{l}$. Let $x^P \in T^P$ be such that $l(x^P) \neq \bar{l}(x^P)$. Denote $x = l(x^P)$ and $\bar{x} = \bar{l}(x^P)$. Let D be defined by $x \in D \in \mathscr{D}(\bar{x})$.

We may assume that there exists $i \in P$ with $x^i \notin D \cup \{\bar{x}\}$. If this is false then the only way $\bar{x}$ can be balancing is if $x^i = \bar{x}$ for all $i \in P$, but then ($\mathscr{U}$) implies that $l(x^P) = \bar{x}$.

Among the points x^j, for $j \in P$ with $x^j \notin D \cup \{\bar{x}\}$, we choose one x^i which is most distant from $\bar{x}$. We define $y^P \in T^P$ as follows. If $j \in P_D(x^P)$ then $y^j = x^j$. Otherwise $y^j \in R\{\bar{x}, x^i\}$ and $d(y^j, \bar{x}) = d(x^j, \bar{x})$; in particular $y^i = x^i$. By repeated applications of ($\mathscr{I}$) we find that $l(y^P) = x$.

Let $\bar{y} = \bar{l}(y^P)$. We claim that $\bar{y} \in R\{\bar{x}, y^i\}$. Indeed, a balancing point for y^P in D would have to be balancing for x^P as well; and if $\bar{y} \notin D \cup R\{\bar{x}, y^i\}$ then all y^j are in the same direction from $\bar{y}$, so $\bar{y}$ cannot be balancing for y^P.

We now compare the situation y^P, for which l indicates x as the solution point, to the situation where all individuals are located at y^i, for which l indicates y^i as the solution point (due to ($\mathscr{U}$)). Repeated applications of ($\mathscr{L}$) permit us to conclude that

$$d(x, y^i) \leqslant \frac{1}{p} \sum_{j=1}^{p} d(y^j, y^i).$$

To evaluate the right-hand side, let D' be defined by $D \subset D' \in \mathscr{D}(\bar{y})$. Then

$$\sum_{j=1}^{p} d(y^j, y^i) = \sum_{j \in P_{D'}(y^P)} \left[d(y^j, \bar{y}) + d(\bar{y}, y^i)\right] + \sum_{j \notin P_{D'}(y^P)} \left[d(\bar{y}, y^i) - d(y^j, \bar{y})\right]$$

$$= \Sigma_{D'}(y^P, \bar{y}) - \Sigma_{\setminus D'}(y^P, \bar{y}) + pd(\bar{y}, y^i) \leqslant pd(\bar{y}, y^i),$$

where the inequality holds because $\bar{y}$ is balancing for y^P. Combining the two inequalities we get $d(x, y^i) \leqslant d(\bar{y}, y^i)$, which is absurd since $\bar{y} \in R\{\bar{x}, y^i\} \subsetneq R\{x, y^i\}$.

The proof of the theorem is now complete.

4. The independence of the axioms. In this section we show that in general none of the three axioms is redundant in the characterization given in the theorem. Only when the infrastructure or the population is degenerate in some sense can a proper subset of the axioms determine the solution.

In all the following propositions T is an infrastructure with the tree property and $P = \{1, \ldots, p\}$ is a population.

PROPOSITION 1. *If T is more than a point then there exists a solution which satisfies ($\mathscr{L}$) and ($\mathscr{I}$) but not ($\mathscr{U}$).*

Indeed, any constant function l: $T^P \to T$ can serve here. Of course, if T is just a point then there is only one solution.

PROPOSITION 2. *If T is more than a point and $p \geqslant 2$ then there exists a solution which satisfies ($\mathscr{U}$) and ($\mathscr{I}$) but not ($\mathscr{L}$).*

Indeed, any projection l_i: $T^P \to T$ (i.e., $l_i(x^P) = x^i$) can serve here. Of course, if $p = 1$ then ($\mathscr{U}$) alone determines the solution.

PROPOSITION 3. *If T is more than a simple curve and $p \geq 3$ then there exists a solution which satisfies ($\mathscr{U}$) and ($\mathscr{L}$) but not ($\mathscr{I}$).*

Here we shall construct a less trivial solution. Let $x^P \in T^P$ be given. We consider the following subset of T:

$$C(T, x^P) = \left\{ x \in T \mid \forall y \in T, d(x, y) \leq \frac{1}{p} \sum_{i=1}^{p} d(x^i, y) \right\}.$$

It is readily seen that this set is connected and closed. Any solution that satisfies ($\mathscr{U}$) and ($\mathscr{L}$) must select from $C(T, \cdot\,)$; this is seen by moving everybody to a point y. In particular, $C(T, x^P) \neq \varnothing$ for all $x^P \in T^P$.

To obtain a nice selection, we define:

$$\text{for } x \in C(T, x^P), \quad \rho(x \mid T, x^P) = \max_{z \in C(T, x^P)} d(x, z),$$

$$\tilde{l}(x^P) = \arg\min_{x \in C(T, x^P)} \rho(x \mid T, x^P).$$

In words, the solution $\tilde{l}$ picks the center of $C(T, x^P)$; that this center is unique is a well-known consequence of the tree property.

It is obvious that $\tilde{l}$ satisfies ($\mathscr{U}$). To verify ($\mathscr{L}$), let x^P and y^P differ only in their ith component, and denote $\delta = d(x^i, y^i)/p$. We establish first that given $x \in C(T, x^P)$ there exists $x' \in C(T, y^P)$ with $d(x, x') \leq \delta$ (and vice-versa, by symmetry). To find such x', we take an arbitrary $z \in C(T, y^P)$, and we may assume that $d(x, z) > \delta$. Let $x' \in R\{x, z\}$ with $d(x, x') = \delta$. We check that $x' \in C(T, y^P)$. Indeed, given $y \in T$, either (a) $x' \in R\{x, y\}$ or (b) $x' \in R\{z, y\}$.

If (a) is true, the argument is:

$$d(x^i, y) \leq d(x^i, y^i) + d(y^i, y) = p\delta + d(y^i, y),$$

$$x \in C(T, x^P) \Rightarrow d(x, y) \leq \frac{1}{p} \sum_{j=1}^{p} d(x^j, y) \leq \frac{1}{p} \sum_{j=1}^{p} d(y^j, y) + \delta,$$

$$x' \in R\{x, y\} \Rightarrow d(x', y) = d(x, y) - \delta \leq \frac{1}{p} \sum_{j=1}^{p} d(y^j, y).$$

If (b) is true, the argument is:

$$z \in C(T, y^P) \Rightarrow d(z, y) \leq \frac{1}{p} \sum_{j=1}^{p} d(y^j, y),$$

$$x' \in R\{z, y\} \Rightarrow d(x', y) \leq d(z, y) \leq \frac{1}{p} \sum_{j=1}^{p} d(y^j, y).$$

Thus x' is as required. We proceed to show that $d(\tilde{l}(x^P), \tilde{l}(y^P)) \leq \delta$. We denote $\tilde{x} = \tilde{l}(x^P)$ and $\tilde{y} = \tilde{l}(y^P)$. We can find a point $x \in C(T, x^P)$ such that $d(\tilde{y}, x) = d(\tilde{y}, \tilde{x}) + \rho(\tilde{x} \mid T, x^P)$. Next, as established above, we can find a point $x' \in C(T, y^P)$

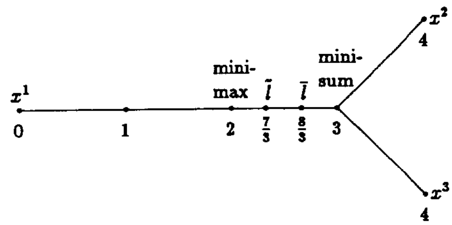

FIGURE 4. A Variety of Solutions (the Numbers Denote Distances from the Left End).

such that $d(x, x') \leqslant \delta$. We have

$$d(\tilde{y}, x) \leqslant d(\tilde{y}, x') + d(x', x) \leqslant \rho(\tilde{y} \mid T, y^P) + \delta,$$

$$d(\tilde{y}, \tilde{x}) = d(\tilde{y}, x) - \rho(\tilde{x} \mid T, x^P) \leqslant \rho(\tilde{y} \mid T, y^P) - \rho(\tilde{x} \mid T, x^P) + \delta.$$

By symmetry, we can also obtain

$$d(\tilde{x}, \tilde{y}) \leqslant \rho(\tilde{x} \mid T, x^P) - \rho(\tilde{y} \mid T, y^P) + \delta.$$

Hence necessarily $d(\tilde{x}, \tilde{y}) \leqslant \delta$, so $(\mathscr{L})$ holds.

Finally, to show that $\tilde{l}$ violates $(\mathscr{I})$, we produce an example of a situation (Figure 4) where $\tilde{l}$ and $\bar{l}$ disagree. A situation of this type is possible whenever T and P are as assumed in Proposition 3. For purposes of comparison and evaluation, we indicate the classical solution points (minisum and minimax; see the Introduction) as well.

The assumptions of Proposition 3 are necessary. If T is just a simple curve then $(\mathscr{I})$ reduces to a tautology; the only way its assumptions can be realized is if $x^P = y^P$. If $p \leqslant 2$ then $C(T, \cdot\,)$ is single valued, hence $(\mathscr{U})$ and $(\mathscr{L})$ determine the solution.

5. Discussion

5.1. *The axiomatic approach.* There is a fundamental difference between our work and the existing literature on the network location problem (e.g., [2], [4]). While others take an objective function as given with the problem and concentrate mostly on the algorithmic aspect of optimization, we concentrate on the conceptual issue of what makes a best location. The axiomatic approach focuses on the principles of equity and consistency involved in settling the typically conflicting claims of the individuals. Thus it makes it possible to argue in favor or against a solution concept, or compare concepts, in terms of the principles satisfied or violated.

To the best of our knowledge, there has been no previous axiomatic treatment of this problem. Axiomatic treatment is quite standard in game theory and economics, in particular in the literature that deals with fairness of distribution mechanisms and public goods. While the facility location problem can be viewed, conceptually, as a special case of these broader issues, nothing in that literature seems to bear directly on our problem.

Some related references, e.g. [9], are mentioned in [10]. These works differ from ours in at least two ways: their subject matter is the continuous location problem (no network structure) and, more significantly, their axioms are formulated in terms of a game abstracted from the location conflict, whereas ours refer directly to the location data. In our view, the direct approach makes it easier to understand what the axioms say about the resolution of the location conflict. Our work bears a closer formal resemblance to works on a continuous model of voting, such as [1] and [7]. The model there corresponds to the special case of our model in which T is an interval. The axioms there are motivated within the context of voting theory, and part of them do not seem attractive in a facility location context.

5.2. *The solution.* The fact that our axioms determine specifically the minimization of the sum of squared distances may have come as a surprise, and it is natural to ask for an intuitive explanation of what drives this result. Although our independence results in §4 indicate that no one (or even two) of the axioms alone is responsible, it seems fair to say that most of the responsibility lies with the Lipschitz axiom ($\mathscr{L}$). Indeed, if one replaces the squares in our solution $\bar{l}$ by any exponent $1 < q < \infty$, one obtains a solution that satisfies ($\mathscr{U}$) and ($\mathscr{I}$); it is ($\mathscr{L}$) that forces $q = 2$. Bearing in mind the well-known fact from statistics that the mean minimizes the sum of squared deviations, the following interpretation suggests itself: the $\bar{l}$ solution is a generalized mean in the same way that the minisum solution is a generalized median.

The solution $\bar{l}$ has not received specific attention in the literature. The only mention we know of occurs in the survey [6], where an algorithm to minimize the sum of squared distances is attributed to Goldman by personal communication. The class of problems of the form $\min \Sigma[d(x^i, x)]^q$, where $1 \leqslant q < \infty$, has been studied in [12], and the more general class of convex minimization problems on tree-networks has been studied in [3]. We could have used facts from these two works to establish Steps 1–3 in the proof of our theorem (but we preferred to keep the proof self-contained). These works also contain algorithms that can be used, in particular, to compute the $\bar{l}$ solution.

The main intuitive appeal of the solution $\bar{l}$ is that it reaches a sensible compromise between the conflicting claims of the individuals. This is best illustrated in Figure 3, where $\bar{l}$ recommends the point whose distances to the endpoints are inversely proportional to the sizes of the subpopulations located at them. By contrast, the minisum solution always recommends an endpoint (except when $m = p/2$) and the minimax solution always recommends the midpoint (except when $m = 0$ or p). The feeling that the $\bar{l}$ outcome in this case is a fair compromise is enhanced by the fact that it corresponds to the Nash solution point of a bargaining problem associated in a natural way with the location problem; this, however, does not generalize beyond the case of Figure 3.

Finally, the solution $\bar{l}$ has some other desirable properties in addition to those stated in the axioms. Two such properties are Pareto optimality (no alternative to the solution point is weakly preferred to it by every individual) and anonymity (the solution point is not affected if the components of the individual locations vector are rearranged). While it is true that these properties are shared by many other solutions, it is nice to know that they are obtained from the system of axioms without being explicitly assumed.

5.3. *Extensions and modifications.* Our theorem has an analogue in the context of the continuous location problem. For this analogue, we replace the infrastructure T by some convex set S in E^n (n-dimensional Euclidean space), and interpret $d(x, y)$ as the Euclidean distance. The result is: there exists a unique solution that satisfies the axioms ($\mathscr{U}$) and ($\mathscr{L}$), and this solution is the function $\bar{l}$: $S^P \to S$ defined by $\bar{l}(x^P) = (1/p)\Sigma_{i=1}^{p} x^i$. We note that axiom ($\mathscr{I}$) is not required here (its continuous analogue is a tautology) and that $\bar{l}(x^P)$ is indeed $\arg\min_{x \in S} \Sigma_{i=1}^{p}[d(x^i, x)]^2$. We omit the proof of this result, which is much easier than that of our main theorem.

A joint special case of the network problem and the continuous problem is when the infrastructure is a closed interval in E^1. We offer a fresh interpretation of our result in this case. Suppose $I = [0, 100]$, with $x \in I$ interpreted as grades. Let $x^1, \ldots, x^p$ be the grades for questions $1, \ldots, p$ in a student's exam (we assume that the exam consists of p questions of equal weight). Under this interpretation, a solution $l: I^P \to I$ is a rule for aggregating the separate grades to obtain a final grade on the exam. Our result is an axiomatic justification for the standard use of the arithmetic mean as the aggregation rule. The axioms involved are ($\mathscr{U}$) and ($\mathscr{L}$), and the result can be rephrased as follows: if one restricts attention to rules that satisfy $l(x, \ldots, x) = x$ and one is interested in minimizing the largest possible effect of a mistake in a single grade on the final grade, one should use the arithmetic mean rule.

It is natural to try to extend our approach and result to general networks (without the tree property). For a general connected infrastructure N, the distance between two points $x, y \in N$ is defined as the length of a shortest route linking x and y (which may not be unique). The concept of direction can be captured by saying that y and z are in the same direction from x if there exist shortest routes linking x to y and z respectively that have a segment in common. Thus, the basic concepts can be adapted, though they lose important properties. Another complication arises from the fact that one must consider multi-valued solutions (i.e., $l(x^P)$ is in general a subset of N), because symmetry considerations preclude single-valued solutions in some situations (e.g., an equilateral triangle with one individual at each vertex). The axioms can be adapted to this more general setup, but they become too demanding. Our feeling is that although the axiomatic approach is in principle applicable to general networks, the current result does not lend itself to such generalization.

We would like to point out three other potential types of continuation and extension of this work:

(a) One may maintain our setup but consider alternative systems of axioms which may characterize other known or new solutions.

(b) One may consider solutions defined over a whole family of infrastructures and/or populations (in our work the infrastructure and the population are fixed). In such a context, one can have axioms that deal with the way the solution reacts to changes in the infrastructure, the arrival of new users, etc.

(c) One may consider generalizations of the basic location problem, such as introducing frequencies of visits at the facility or increasing the number of facilities to be located.

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