Int J Game Theory (2001) 29:543–553



# The comparability of the classical and the Mas-Colell bargaining sets

**Ron Holzman***

Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel
(email: holzman@techunix.technion.ac.il)

Received: April 2000/Revised version: November 2000

**Abstract.** The Mas-Colell bargaining set is shown to contain the classical bargaining set for superadditive NTU games satisfying the nonlevelness condition. Without superadditivity this is no longer true, but in the TU case the containment still holds for the closure of the Mas-Colell bargaining set.

**Key words:** TU games, NTU games, bargaining sets, containment relations

## 1. Introduction

Aumann and Maschler (1964) introduced a variety of solution concepts for cooperative games, called bargaining sets. These concepts have a common underlying logic of objections and counter-objections: a payoff vector is in the bargaining set if for every objection to it there is a counter-objection. One of the variants, denoted by $\mathscr{M}_1^{(i)}$, was explicitly defined by Davis and Maschler (1963, 1967), and was further studied by these and many other researchers. It emerged as the most important variant, and we will refer to it as the classical bargaining set.

Mas-Colell (1989) proposed a new variant of a bargaining set. He was mainly concerned with atomless exchange economies, but he also defined his bargaining set for general cooperative games. We will refer to the latter as the Mas-Colell bargaining set, and denote it by $\mathscr{MB}$. Discussing the relation of his bargaining set to the classical one, Mas-Colell wrote:

* Work partly done while the author was visiting the Department of Mathematics, Massachusetts Institute of Technology. Correspondence with Michael Maschler is gratefully acknowledged. The author also thanks an anonymous referee for a suggestion concerning the presentation.

> "Strictly speaking the relation to the Aumann-Maschler Bargaining Set is one of non-comparability. This is because on the one hand we make counterobjecting easier (no member of the objection coalition is excluded a priori from belonging to a counterobjecting group) but on the other we make it a bit harder by requiring that at least one of the inequalities defining the counterobjection be strict."

Vohra (1991), who further investigated the Mas-Colell bargaining set, made similar comments.

Although, as pointed out in the above quote, the definitions of objections and counter-objections for $\mathscr{M}_1^{(i)}$ and $\mathscr{MB}$ are non-comparable, we will show here that the resulting bargaining sets *are* in fact comparable for a large class of games. Indeed, we will prove that the Mas-Colell bargaining set contains the classical one for every superadditive game. This holds both in the transferable utility (TU) case and in the non-transferable utility (NTU) case, though for the latter we need to assume nonlevelness. If the game is not superadditive, $\mathscr{M}_1^{(i)}$ may contain points which are not in $\mathscr{MB}$, but in the TU case we will show that these points can be approximated arbitrarily well by points in $\mathscr{MB}$.

We note that the classical bargaining set is known to be non-empty for TU games with a non-empty set of imputations (Davis and Maschler 1963, 1967; Peleg 1963, 1967). Thus our results provide a new non-emptiness proof for the Mas-Colell bargaining set in the TU case. Earlier proofs were based on the fact that it contains the prekernel (Mas-Colell 1989, Vohra 1991) and the least core (Einy *et al.* 1999). A direct topological proof of non-emptiness was given by Vohra (1991).

The situation in the NTU case is not as good. Non-superadditive NTU games with an empty Mas-Colell bargaining set are known to exist. The non-emptiness of $\mathscr{MB}$ for superadditive NTU games is an open problem (some sufficient conditions were given by Vohra (1991)). Unfortunately, our results do not help in this respect, since in the NTU case $\mathscr{M}_1^{(i)}$ may be empty even for superadditive games.

## 2. TU games

A *TU game* is a pair $(N, v)$, where $N$, the set of *players*, is a non-empty finite set, and $v$, the *characteristic function*, is a function from the power set $\mathscr{P}(N)$ to $\Re$ satisfying $v(\emptyset) = 0$. We refer to any $S \in \mathscr{P}(N)$, that is, $S \subseteq N$, as a *coalition*. The number $v(S)$ is thought of as the worth of the coalition $S$. The game $(N, v)$ is *superadditive* if for every two disjoint coalitions $S$ and $T$ we have $v(S \cup T) \geq v(S) + v(T)$. It is *zero-monotonic* if for every coalition $S$ and every player $i \in N \backslash S$ we have $v(S \cup \{i\}) \geq v(S) + v(\{i\})$.

For a non-empty coalition $S$, we denote by $\Re^S$ the $|S|$-dimensional Euclidean space with coordinates indexed by the players in $S$. If $x \in \Re^S$ and $T \subseteq S$, we write $x(T)$ for $\sum_{i \in T} x_i$. The set of *preimputations* in $(N, v)$ is the set:

$$\mathscr{X}^0 = \mathscr{X}^0(N, v) = \{x \in \Re^N \mid x(N) = v(N)\}$$

The set of *imputations* in $(N, v)$ is the set:

$$\mathscr{X} = \mathscr{X}(N, v) = \{x \in \Re^N \mid x(N) = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

We remark that in more generality, preimputations and imputations are defined with respect to a coalition structure. Since the Mas-Colell bargaining set was defined only with respect to the coalition structure $\{N\}$, here we restrict attention to this case.

For a preimputation $x$ and a coalition $S$ in a game $(N, v)$, the *excess* of $S$ at $x$ is defined by

$$e(S, x) = v(S) - x(S).$$

The *core* of $(N, v)$ is the set:

$$\mathscr{C} = \mathscr{C}(N, v) = \{x \in \mathscr{X} \mid e(S, x) \leq 0 \text{ for all } S \subseteq N\}$$

We proceed now to define the classical bargaining set. Let $(N, v)$ be a game, let $x$ be an imputation, and let $k$ and $l$ be distinct players. An *objection* of $k$ against $l$ at $x$ is a pair $(C, y)$, where $C$ is a coalition containing $k$ but not $l$, and $y$ is in $\Re^C$ and satisfies

$$y(C) = v(C), \tag{1}$$

$$y_i > x_i \quad \text{for all } i \in C. \tag{2}$$

Let $(C, y)$ be an objection of $k$ against $l$ at $x$. A *counter-objection* to this objection is a pair $(D, z)$, where $D$ is a coalition containing $l$ but not $k$, and $z$ is in $\Re^D$ and satisfies

$$z(D) = v(D), \tag{3}$$

$$z_i \geq y_i \quad \text{for all } i \in D \cap C, \tag{4}$$

$$z_i \geq x_i \quad \text{for all } i \in D \backslash C. \tag{5}$$

An objection is *justified* if there is no counter-objection to it. The *classical bargaining set* of $(N, v)$ is the set:

$$\mathscr{M}_1^{(i)} = \mathscr{M}_1^{(i)}(N, v) = \{x \in \mathscr{X} \mid \text{no player has a justified objection at } x \text{ against any other player}\}$$

Next, we define the Mas-Colell bargaining set. Let $(N, v)$ be a game and let $x$ be a preimputation. An *objection* at $x$ is a pair $(C, y)$, where $C$ is a nonempty coalition and $y$ is in $\Re^C$ and satisfies

$$y(C) = v(C), \tag{6}$$

$$y_i \geq x_i \quad \text{for all } i \in C, \tag{7}$$

and at least one of the inequalities in (7) is strict. Let $(C, y)$ be an objection at $x$. A *counter-objection* to this objection is a pair $(D, z)$, where $D$ is a nonempty coalition and $z$ is in $\Re^D$ and satisfies

$$z(D) = v(D), \tag{8}$$

$$z_i \geq y_i \quad \text{for all } i \in D \cap C, \tag{9}$$

$$z_i \geq x_i \quad \text{for all } i \in D \backslash C, \tag{10}$$

and at least one of the inequalities in (9) or (10) is strict. An objection is *justified* if there is no counter-objection to it. The *Mas-Colell bargaining set* of $(N, v)$ is the set:

$$\mathcal{MB} = \mathcal{MB}(N, v) = \{x \in \mathcal{X}^0 \mid \text{no non-empty coalition has a justified objection at } x\}$$

We are ready to state our first result.

**Theorem 2.1.** *Let $(N, v)$ be a superadditive TU game. Then*

$$\mathcal{M}_1^{(i)}(N, v) \subseteq \mathcal{MB}(N, v).$$

We do not give here a direct proof of Theorem 2.1, because it is a particular case of Theorem 3.1, concerning NTU games, which we prove in the next section. Alternatively, one can deduce Theorem 2.1 from a result of Solymosi (1999). For a game $(N, v)$ and a preimputation $x$, he defined the *maximal excess game* at $x$ to be the game $(N, w_x)$ whose characteristic function is

$$w_x(S) = \max\{e(T, x) \mid T \subseteq S\}.$$

He proved (Solymosi 1999, Theorem 1) that if $(N, v)$ is superadditive and $x \in \mathcal{M}_1^{(i)}(N, v)$ then $x \in \mathcal{C}(N, v)$ if and only if $\mathcal{C}(N, w_x) \neq \emptyset$. Solymosi did not consider the Mas-Colell bargaining set, but the definition of the latter can be naturally cast in terms of the cores of the maximal excess games. Indeed, we have the following.

**Observation 2.2.** *Let $(N, v)$ be a TU game, and let $x$ be a preimputation in $(N, v)$ such that $x \notin \mathcal{C}(N, v)$. Then a pair $(C, y)$ is a justified objection at $x$ if and only if $e(C, x) = w_x(N)$ and the vector $u$ defined by*

$$u_i = \begin{cases} y_i - x_i & \text{if } i \in C, \\ 0 & \text{if } i \in N \backslash C, \end{cases} \tag{11}$$

*satisfies $u \in \mathcal{C}(N, w_x)$.*

*Proof.* Suppose first that $(C, y)$ is a justified objection at $x$. The absence of a counter-objection implies that for every non-empty coalition $D$ we have

$$y(D \cap C) + x(D \backslash C) \geq v(D). \tag{12}$$

Upon subtracting $x(D)$ from both sides, we obtain $u(D) \geq e(D, x)$. As this holds for every $D$, and $u$ has nonnegative components (by (7)), it follows that $u(S) \geq w_x(S)$ for every coalition $S$. The fact that

$$u(N) = u(C) = y(C) - x(C) = v(C) - x(C) = e(C, x)$$

completes the proof that $u \in \mathscr{C}(N, w_x)$ and $e(C, x) = w_x(N)$.

Conversely, suppose that $e(C, x) = w_x(N)$ and $u \in \mathscr{C}(N, w_x)$. As $x \notin \mathscr{C}(N, v)$, we have $w_x(N) > 0$ and it follows from $u \in \mathscr{C}(N, w_x)$ that $C \neq \emptyset$ and at least one of the $C$-components of $u$ is positive, while the other components are nonnegative by the nonnegativity of $w_x$. To establish that $(C, y)$ is an objection at $x$, we need only note further that

$$u(C) = u(N) = w_x(N) = e(C, x) = v(C) - x(C)$$

and hence $y(C) = v(C)$. Finally, no counter-objection to $(C, y)$ exists, because for every $D$ we have $u(D) \geq w_x(D) \geq e(D, x)$ which implies (12) upon addition of $x(D)$.

**Corollary 2.3.** *Let $(N, v)$ be a TU game, and let $x$ be a preimputation in $(N, v)$. Then $x \in \mathscr{MB}(N, v) \backslash \mathscr{C}(N, v)$ if and only if $\mathscr{C}(N, w_x) = \emptyset$.*

*Proof.* Suppose first that $x \in \mathscr{MB}(N, v) \backslash \mathscr{C}(N, v)$, but $\mathscr{C}(N, w_x) \neq \emptyset$. Let $C$ be a coalition such that $e(C, x) = w_x(N)$, and let $u$ be an element of $\mathscr{C}(N, w_x)$. Then $u_i = 0$ for $i \in N \backslash C$, and we can find $y \in \Re^C$ which induces $u$ according to (11). It follows from Observation 2.2 that $(C, y)$ is a justified objection at $x$, contradicting the assumption that $x \in \mathscr{MB}(N, v)$.

Conversely, suppose that $\mathscr{C}(N, w_x) = \emptyset$. Then $w_x$ is not the zero game, and therefore $x \notin \mathscr{C}(N, v)$. Furthermore, by Observation 2.2 there can be no justified objection at $x$, and so $x \in \mathscr{MB}(N, v)$.

This corollary, together with the fact that both bargaining sets contain the core, renders Theorem 2.1 and Solymosi's Theorem 1 equivalent.

The inclusion in Theorem 2.1 may be strict. In fact, $\mathscr{MB}$ may be much larger than $\mathscr{M}_1^{(i)}$, as illustrated by the following simple majority example.

**Example 2.1.**
Let $N = \{1, 2, 3\}$ and let

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 2, \\ 0 & \text{if } |S| \leq 1. \end{cases}$$

One can verify that

$$\mathscr{M}_1^{(i)} = \{(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3})\}$$

while

$$\mathscr{MB} = \{(x_1, x_2, x_3) | x_1 + x_2 + x_3 = 1, x_i < \tfrac{1}{2} \text{ for } i = 1, 2, 3\}.$$

The difference between the two bargaining sets arises because, when an objection is made via a two-player coalition $S$, in the case of $\mathscr{MB}$ the third player is allowed to recruit any of the two members of $S$ for a counter-objection, whereas for the purposes of $\mathscr{M}_1^{(i)}$ one of the members of $S$ is designated as the objector and is not available for counter-objecting.

The conclusion of Theorem 2.1 need not hold for non-superadditive games. In fact, the following example borrowed from Solymosi (1999) shows that the superadditivity assumption in Theorem 2.1 cannot be weakened to zero-monotonicity.

**Example 2.2.**
Let $N = \{1, 2, 3, 4\}$ and let

$$v(S) = \begin{cases} 3 & \text{if } S = \{1, 2\}, \\ 0 & \text{if } |S| = 1, \\ |S| & \text{otherwise.} \end{cases}$$

One can verify that the imputation $x = (1, 1, 1, 1)$ is in $\mathcal{M}_1^{(i)}$ but not in $\mathcal{MB}$. The reason for this is that an objection of 1 or 2 against 3 or 4 can be countered via the coalition $\{3, 4\}$, since only weak inequalities are required in (5). In the sense of Mas-Colell, however, the coalition $\{1, 2\}$ has a justified objection, because a counter-objection requires at least one strict inequality. It is worth noting that for arbitrarily small $\varepsilon > 0$, the imputation $(1 + \varepsilon, 1 + \varepsilon, 1 - \varepsilon, 1 - \varepsilon)$ is in $\mathcal{MB}$. Thus, although $x$ is not in $\mathcal{MB}$, it is in its closure $\overline{\mathcal{MB}}$.

Our next result shows that the observation made in the analysis of the last example is valid in general.

**Theorem 2.4.** *Let $(N, v)$ be a TU game. Then*

$$\mathcal{M}_1^{(i)}(N, v) \subseteq \overline{\mathcal{MB}(N, v)}.$$

For the proof of Theorem 2.4, we need another corollary of Observation 2.2. Given a preimputation $x$ in a game $(N, v)$, we denote by $\mathrm{Mex}(x)$ the collection of coalitions which attain the maximum excess at $x$, that is,

$$\mathrm{Mex}(x) = \{S \subseteq N \mid e(S, x) = w_x(N)\}.$$

**Corollary 2.5.** *Let $(N, v)$ be a TU game, let $x$ be a preimputation in $(N, v)$, and let $(C, y)$ be a justified objection at $x$ (in the sense of Mas-Colell). Then:*
(a) $C \in \mathrm{Mex}(x)$.
(b) $\bigcap \mathrm{Mex}(x) \neq \emptyset$.

*Proof.* Part (a) follows directly from Observation 2.2. To obtain (b), let $k$ be a member of $C$ for whom the inequality in (7) is strict. Then $k$ belongs to every coalition in $\mathrm{Mex}(x)$. Indeed, if $S$ is any coalition not containing $k$, and $u$ is the vector defined by (11), then we have

$$e(C, x) = v(C) - x(C) = y(C) - x(C) > u(C \backslash \{k\}) \geq u(S)$$
$$\geq w_x(S) \geq e(S, x)$$

which implies that $S \notin \mathrm{Mex}(x)$.

*Proof of Theorem 2.4.* Suppose, for contradiction, that $x \in \mathscr{M}_1^{(i)} \setminus \overline{\mathscr{MB}}$. Since $x \notin \mathscr{MB}$, there is a justified objection at $x$, and so by Corollary 2.5(b) there exists a player, say $k$, who belongs to every coalition in $\mathrm{Mex}(x)$.

Define the preimputation $x^\varepsilon$, for $\varepsilon \in \Re$, by

$$x_i^\varepsilon = \begin{cases} x_i - \varepsilon & \text{if } i \neq k, \\ x_k + (n-1)\varepsilon & \text{if } i = k. \end{cases}$$

(Here $n = |N|$.) Choose $\varepsilon > 0$ small enough so that

$$x^\varepsilon \notin \mathscr{MB} \tag{13}$$

and

$$\mathrm{Mex}(x^\varepsilon) \subseteq \mathrm{Mex}(x). \tag{14}$$

By (13), there exists a justified objection $(C, y^\varepsilon)$ at $x^\varepsilon$. Applying Corollary 2.5(a) to $x^\varepsilon$, it follows that $C \in \mathrm{Mex}(x^\varepsilon)$. By (14) and the choice of $k$, this implies that $C$ contains $k$. Clearly $C \neq N$, so let $l$ be a member of $N \setminus C$.

Construct an objection $(C, y)$ of $k$ against $l$ at $x$ by

$$y_i = \begin{cases} y_i^\varepsilon + \dfrac{n - \frac{3}{2}}{|C| - 1}\varepsilon & \text{if } i \in C \setminus \{k\}, \\[2ex] y_k^\varepsilon - (n - \frac{3}{2})\varepsilon & \text{if } i = k. \end{cases}$$

Note, first, that this is well-defined: $|C| > 1$, since otherwise $C = \{k\}$ and (6, 7) imply $v(\{k\}) \geq x_k^\varepsilon > x_k$, which contradicts the fact that $x \in \mathscr{M}_1^{(i)} \subseteq \mathscr{X}$. It is easy to verify that properties (6, 7) of $y^\varepsilon$ with respect to $x^\varepsilon$ imply properties (1,2) of $y$ with respect to $x$, so that $(C, y)$ is indeed an objection of $k$ against $l$ at $x$.

Since $x \in \mathscr{M}_1^{(i)}$, there exists a counter-objection $(D, z)$ to $(C, y)$. As $k \notin D$, this is also a counter-objection to $(C, y^\varepsilon)$ at $x^\varepsilon$ in the sense of Mas-Colell, because $z_i \geq y_i > y_i^\varepsilon$ for $i \in D \cap C$ and $z_i \geq x_i > x_i^\varepsilon$ for $i \in D \setminus C$. This, however, contradicts the fact that $(C, y^\varepsilon)$ is justified.

The two bargaining sets differ also in their treatment of individual rationality: $\mathscr{M}_1^{(i)}$ is defined as a subset of the set of imputations $\mathscr{X}$, while $\mathscr{MB}$ is defined as a subset of the set of preimputations $\mathscr{X}^0$. We comment briefly on the effect of changes in this respect on our results. If $\mathscr{X}$ is replaced by $\mathscr{X}^0$ in the definition of $\mathscr{M}_1^{(i)}$, one obtains the prebargaining set $\mathscr{PM}_1^{(i)}$. Our results remain true when $\mathscr{M}_1^{(i)}$ is replaced by $\mathscr{PM}_1^{(i)}$. If $\mathscr{X}^0$ is replaced by $\mathscr{X}$ in the definition of $\mathscr{MB}$, one obtains the bargaining set studied by Vohra (1991), which we denote by $\mathscr{IRMB}$. It is obvious that Theorem 2.1 remains true for this variant, since $\mathscr{M}_1^{(i)} \subseteq \mathscr{X}$. The analog of Theorem 2.4 requires more care. The proof above may be adapted to show that if $(N, v)$ is zero-monotonic then $\mathscr{M}_1^{(i)} \subseteq \overline{\mathscr{IRMB}}$. Without zero-monotonicity, however, this may fail, as illustrated by the following example.

**Example 2.3.**
Let $N = \{1, 2, 3\}$ and let

$$v(S) = \begin{cases} 1 & \text{if } S = \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

In this example $\mathscr{X} = \{(0, 0, 0)\}$. The unique imputation is in $\mathscr{M}_1^{(i)}$ but not in $\mathscr{MB}$. The latter implies that $\mathscr{IRMB}$ is empty, and so is its closure.

## 3. NTU games

An *NTU game* is a pair $(N, V)$, where $N$, the set of *players*, is a non-empty finite set, and $V$, the *characteristic function*, assigns to every non-empty coalition $S \subseteq N$ a non-empty subset $V(S) \subseteq \Re^S$ satisfying the following properties:

*Zero-normalization*: For every $i \in N$,

$$V(\{i\}) = \{x_i | x_i \leq 0\}.$$

*Comprehensiveness*: For every $\varnothing \neq S \subseteq N$,

$$\text{if } x, y \in \Re^S, \quad y \in V(S) \quad \text{and} \quad x \leq y, \quad \text{then } x \in V(S).$$

*Compactness*: For every $\varnothing \neq S \subseteq N$,

$$V(S) \cap \Re^S_+ \quad \text{is compact.}$$

(We have used the notation $x \leq y$ for $x, y \in \Re^S$ meaning $x_i \leq y_i$ for every $i \in S$. We will use the notation $x \ll y$ to mean $x_i < y_i$ for every $i \in S$.)

The set $V(S)$ is interpreted as the set of utility vectors that the coalition $S$ can achieve for its members. Its subset consisting of the weakly efficient utility vectors is denoted by $V^0(S)$ and defined by

$$V^0(S) = \{x \in V(S) \,|\, \text{there is no } y \in V(S) \text{ such that } x \ll y\}.$$

The game $(N, V)$ is *nonlevel* if it satisfies the following additional condition:

*Nonlevelness*: For every $\varnothing \neq S \subseteq N$,

$$\text{if } x, y \in V^0(S) \quad \text{and} \quad x \leq y \quad \text{then } x = y.$$

For two disjoint non-empty coalitions $S$ and $T$ and two vectors $x \in \Re^S$ and $y \in \Re^T$, we denote by $(x, y)$ the vector in $\Re^{S \cup T}$ whose $S$-components are as in $x$ and $T$-components are as in $y$. The game $(N, V)$ is *superadditive* if for every two disjoint non-empty coalitions $S$ and $T$ and any two vectors $x \in V(S)$ and $y \in V(T)$ we have $(x, y) \in V(S \cup T)$. It is *zero-monotonic* if the latter holds whenever $T$ is a one-player coalition.

The set of *preimputations* in $(N, V)$ is the set $\mathscr{X}^0 = V^0(N)$. The set of *imputations* in $(N, V)$ is the set $\mathscr{X} = V^0(N) \cap \Re^N_+$.

The definitions of the *classical bargaining set* $\mathscr{M}_1^{(i)}(N, V)$ and the *Mas-Colell bargaining set* $\mathscr{MB}(N, V)$ for NTU games are identical to the corresponding ones for TU games, except that the condition $y(C) = v(C)$ in (1) and (6) is replaced by $y \in V(C)$ and similarly the condition $z(D) = v(D)$ in (3) and (8) is replaced by $z \in V(D)$.

A TU game $(N, v)$ which is zero-normalized (that is, $v(\{i\}) = 0$ for all $i \in N$) may be identified with the nonlevel NTU game $(N, V)$ defined by

$$V(S) = \{x \in \Re^S \mid x(S) \leq v(S)\}$$

for $\emptyset \neq S \subseteq N$. The corresponding concepts for TU and NTU games defined above coincide for this identification. Moreover, zero-normalization entails no loss of generality. Hence the following result is a generalization of Theorem 2.1.

**Theorem 3.1.** *Let $(N, V)$ be a nonlevel superadditive NTU game. Then*

$$\mathcal{M}_1^{(i)}(N, V) \subseteq \mathcal{MB}(N, V).$$

*Proof.* Suppose, for contradiction, that $x \in \mathcal{M}_1^{(i)} \setminus \mathcal{MB}$. Let $(C, y)$ be a justified objection (in the sense of Mas-Colell) at $x$, chosen so that $C$ is as large as possible among all justified objections. Let $k$ be a member of $C$ for whom the inequality in (7) is strict. The nonlevelness of $V(N)$ implies that $C \neq N$. Let $l$ be a player in $N \setminus C$.

By the comprehensiveness and nonlevelness of $V(C)$, we can modify $y$, while staying in $V(C)$, so that its $k$-component will decrease (but remain above $x_k$) and the other components will increase. Doing this, we obtain a vector $\tilde{y} \in V(C)$ satisfying

$$\tilde{y}_k > x_k,$$

$$\tilde{y}_i > y_i \quad \text{for all } i \in C \setminus \{k\}.$$

Clearly, $(C, \tilde{y})$ is an objection of $k$ against $l$ at $x$. Since $x \in \mathcal{M}_1^{(i)}$, there exists a counter-objection $(D, z)$ to $(C, \tilde{y})$. As $k \notin D$, we have

$$z_i \geq \tilde{y}_i > y_i \quad \text{for all } i \in D \cap C,$$

$$z_i \geq x_i \quad \text{for all } i \in D \setminus C.$$

Thus, $(D, z)$ satisfies all the requirements to be a counter-objection to $(C, y)$ in the sense of Mas-Colell, except perhaps the need for at least one strict inequality in (9) or (10). Since $(C, y)$ is justified, and the inequalities in (9), if any, are strict, it must be the case that $D \cap C = \emptyset$. But then, by superadditivity, $(C \cup D, (y, z))$ is a justified objection at $x$, contradicting the choice of $C$ to be maximal.

The following simple example shows that nonlevelness is needed in Theorem 3.1.

**Example 3.1.**
Let $N = \{1, 2\}$ and let

$$V(S) = \begin{cases} \{(x_1, x_2) | x_1, x_2 \le 1\} & \text{if } S = \{1,2\}, \\ \{x_i | x_i \le 0\} & \text{if } S = \{i\}. \end{cases}$$

The set of imputations in this game is the union of the two segments joining $(1,0)$ and $(0,1)$ to $(1,1)$. All these imputations are in $\mathscr{M}_1^{(i)}$, but only $(1,1)$ is in $\mathscr{MB}$.

Our next example shows that Theorem 2.4 cannot be extended to NTU games.

**Example 3.2.**
Let $N = \{1,2,3,4,5,6\}$ and let

$$V(S) = \begin{cases} \{x \in \Re^N \mid x(N) \le 24\} & \text{if } S = N, \\ \{x \in \Re^S \mid 6x_i + x_j + x_k + 6x_l \le 74\} & \text{if } S = \{i,j,k,l\} \text{ and } (i,j,k,l) \in \{(1,2,3,4), (3,4,5,6), (5,6,1,2)\}, \\ \{x \in \Re^S \mid x(S) \le 8\} & \text{if } S \in \{\{1,2\},\{3,4\},\{5,6\}\}, \\ \{x \in \Re^S \mid x(S) \le 0\} & \text{otherwise.} \end{cases}$$

The imputation $x = (4,4,4,4,4,4)$ is in $\mathscr{M}_1^{(i)}$, since objections are possible only via one of the three powerful four-player coalitions, but the complement of such a coalition provides a counter-objection. On the other hand, $x \notin \overline{\mathscr{MB}}$. Indeed, suppose that $\tilde{x}$ is an imputation and $\|\tilde{x} - x\|_\infty < 1/7$. Assume also, w.l.o.g., that $\tilde{x}_5 + \tilde{x}_6 \ge 8$. Then $(\{1,2,3,4\}, (5,7,7,5))$ is an objection at $\tilde{x}$, which is justified: the coalition $\{5,6\}$ cannot provide a counter-objection with a strict inequality, and a counter-objection by one of the other powerful four-player coalitions, say $\{3,4,5,6\}$, would require $z_3 \ge 7, z_4 \ge 5, z_5 > 4 - 1/7$, $z_6 > 4 - 1/7$ which is impossible for $z \in V(\{3,4,5,6\})$.

We note that this example is a hyperplane game, that is, the sets $V^0(S)$ are hyperplanes. It is perhaps remarkable that the conclusion of Theorem 2.4 fails for a game in this class, which is in some sense close to the class of TU games. Another remark is that this example may be modified to become zero-monotonic (while losing hyperplanarity) without affecting the above analysis.

In the NTU case, the classical bargaining set may be empty even for superadditive games. This led Asscher (1976, 1977) to introduce two related solution concepts: the ordinal bargaining set $\mathscr{M}^o$ and the cardinal bargaining set $\mathscr{M}^c$ (the latter was defined only for nonlevel NTU games satisfying the additional condition that $V(S)$ is convex for every $\emptyset \neq S \subseteq N$). We do not state the definitions here; we only mention that both of these concepts employ the notion of justified objection as defined for the classical bargaining set, but allow for the mutual waiving of justified objections along cycles of players.

These bargaining sets satisfy the inclusions $\mathscr{M}_1^{(i)} \subseteq \mathscr{M}^c \subseteq \mathscr{M}^o$, and $\mathscr{M}^o$ and $\mathscr{M}^c$ are non-empty for every NTU game with a non-empty set of imputations for which they are defined. It would be nice to be able to replace $\mathscr{M}_1^{(i)}$ in Theorem 3.1 by one of these larger bargaining sets, and thereby obtain sufficient conditions for the non-emptiness of $\mathscr{MB}$. The following example indicates that this cannot be done.

**Example 3.3.**
Let $N = \{1,2,3,4,5,6\}$ and let

$$V(S) = \begin{cases} \{x \in \Re^N \mid x(N) \leq 24\} & \text{if } S = N, \\ \{x \in \Re^S \mid x_i \leq 5, x_j \leq 7, x_k \leq 7, x_l \leq 5, & \text{if } S = \{i,j,k,l,m\}, \\ \quad x_m \leq 0\} & (i,j,k,l) \in \{(1,2,3,4), \\ & (3,4,5,6),(5,6,1,2)\} \\ & \text{and } m \in N \backslash \{i,j,k,l\}, \\ \{x \in \Re^S \mid x_i \leq 5, x_j \leq 7, x_k \leq 7, x_l \leq 5\} & \text{if } S = \{i,j,k,l\} \text{ and} \\ & (i,j,k,l) \in \{(1,2,3,4), \\ & (3,4,5,6),(5,6,1,2)\}, \\ \{x \in \Re^S \mid x_i \leq 0 \text{ for all } i \in S\} & \text{otherwise.} \end{cases}$$

The game $(N, V)$ violates nonlevelness, but we can modify it slightly to obtain a game $(N, \tilde{V})$ which is nonlevel, superadditive and has convex $\tilde{V}(S)$ for every $\emptyset \neq S \subseteq N$, while preserving the symmetries of $(N, V)$. For such $(N, \tilde{V})$, one can verify that the imputation $(4,4,4,4,4,4)$ is in $\mathscr{M}^c$ but not in $\mathscr{MB}$ (in fact, not even in $\overline{\mathscr{MB}}$).

We remark that in each of the last two examples, the symmetries of the game make the imputation $(4,4,4,4,4,4)$ a natural outcome. From this point of view, it seems that the classical bargaining set in Example 3.2 and the related ordinal and cardinal bargaining sets in Example 3.3 perform better than the Mas-Colell bargaining set.

## References

Asscher N (1976) An ordinal bargaining set for games without side payments. Math Oper Res 1:381–389

Asscher N (1977) A cardinal bargaining set for games without side payments. Int J Game Theory 6:87–114

Aumann RJ, Maschler M (1964) The bargaining set for cooperative games. In: Dresher M, Shapley LS, Tucker AW (eds.) Advances in game theory. Princeton University Press, Princeton, pp. 443–476

Davis M, Maschler M (1963) Existence of stable payoff configurations for cooperative games. Bull Amer Math Soc 69:106–108

Davis M, Maschler M (1967) Existence of stable payoff configurations for cooperative games. In: Shubik M (ed.) Essays in mathematical economics in honor of Oskar Morgenstern. Princeton University Press, Princeton, pp. 39–52

Einy E, Holzman R, Monderer D (1999) On the least core and the Mas-Colell bargaining set. Games Econ Behav 28:181–188

Mas-Colell A (1989) An equivalence theorem for a bargaining set. J Math Econ 18:129–139

Peleg B (1963) Existence theorem for the bargaining set $\mathscr{M}_1^{(i)}$. Bull Amer Math Soc 69:109–110

Peleg B (1967) Existence theorem for the bargaining set $\mathscr{M}_1^{(i)}$. In: Shubik M (ed.) Essays in mathematical economics in honor of Oskar Morgenstern. Princeton University Press, Princeton, pp. 53–56

Solymosi T (1999) On the bargaining set, kernel and core of superadditive games. Int J Game Theory 28:229–240

Vohra R (1991) An existence theorem for a bargaining set. J Math Econ 20:19–34