JOURNAL OF ECONOMIC THEORY **68**, 200–211 (1996)
ARTICLE NO. 0010

Core and Stable Sets of Large Games Arising in Economics

EZRA EINY*

Department of Economics, Ben-Gurion University, Beer Sheva, Israel

RON HOLZMAN*,†

Department of Mathematics, Technion—Israel Institute of Technology, Haifa, Israel

DOV MONDERER*,†

Faculty of Industrial Engineering and Management, Technion—Israel Institute of Technology, Haifa, Israel

AND

BENYAMIN SHITOVITZ*,‡

Department of Economics, University of Haifa, Haifa, Israel

Received June 15, 1994; revised November 26, 1994

It is shown that the core of a non-atomic glove-market game which is defined as the minimum of finitely many non-atomic probability measures is a von Neumann–Morgenstern stable set. This result is used to characterize some stable sets of large games which have a decreasing returns to scale property. We also study exact non-atomic glove-market games. In particular we show that in a glove-market game which consists of the minimum of finitely many mutually singular non-atomic measures, the core is a von Neumann-Morgenstern stable set iff the game is exact. *Journal of Economic Literature* Classification Numbers: C70, C71, D24.

* We thank Sergiu Hart, Yossi Greenberg, and Abraham Neyman for very helpful discussions concerning this work.

† This author's research was supported by the Tragovnik Research Fund and by the fund for the promotion of research at the Technion.

‡ Part of this work was done while this author was visiting the Center for Rationality and Interactive Decision Theory, the Hebrew University and the Department of Economics at McGill University.

0022-0531/96 $18.00

1. Introduction

Von Neumann and Morgenstern, in their seminal book, *Theory of Games and Economic Behavior* (see [17]), introduced the stable set known as "the solution" of a cooperative game. They believed that this solution concept should be the main tool for the analysis of economic environments. Since then stable sets have been studied intensively, especially for cooperative games with a finite set of players (for a comprehensive survey on this topic, see [8]) Unfortunately there is no general theory for stable sets. This can be explained by the following quotation from Aumann [2]: "What has prevented the theory from being successful is that it is so difficult to work with. Finding stable sets involves a new tour de force of mathematical reasoning for each game or class of games that is considered. Other than a small number of very elementary truisms (e.g., that the core is contained in every stable set), there is no theory, no tools, certainly no algorithms."

Stable sets of finite market games were studied in [14] and of market games with a continuum of players in [7]. In this work we study stable sets and the core of some transferable utility games with a continuum of players which arise in economic applications. Note that, in contrast, the core has been successfully applied to economics, and has had a considerable impact on game theory and its economic applications. One of its attractions is perhaps its close relationship to competitive allocations in atomless markets [1].

The case when the core is also a stable set is clearly desirable. This has been shown to be the case for finite convex games [11, 15]. Nevertheless, for atomless market games it has been generally accepted that the core, being generically a single allocation (the competitive allocation by the equivalence theorem), rarely constitutes a stable set, though of course it should be contained in it. Here, somewhat surprisingly, we prove that the core of an exact glove-market game with a continuum of owners is a von Neumann and Morgenstern solution, where by "exact" we mean that there are equal aggregate amounts of the different types of gloves in the market. Thus, the core, in addition to its internal stability property, has also an external stability property: that is, for every allocation outside the core of the game there exists an allocation in the core that blocks it.

The glove-market game is the game theoretical embodiment of the case where we have complementarity in consumption goods. The usual example is thus of left- and right-hand gloves, where we need both a left-hand glove and a right-hand glove in order to consume. This model is well known and its core as well as other solution concepts has been extensively investigated (see [4, 12 and 16]). In [4] the authors also deal with an interesting atomless linear production model that can be represented by a glove-market game. Thus, our main result covers also exact atomless linear production models.

In this paper, the game defined as the minimum of finitely many measures is called a glove-market game. Moreover, one can define (as we do in the proof of the main theorem) a pure exchange market in which the initial bundle densities are the densities of the measures (i.e., the densities of the different types of gloves), and the utilities are the minimum of the consumed densities of gloves of the different types. (For mathematical convenience we have also added the sum of the consumed densities to the utility function in order to render the utility strictly monotonic.) Using this construction, we can prove the main result, provided that the distributions of the different types of gloves—the measures—are probability atomless measures. This result may not hold for exact glove-market games with finitely many players (see Example 2.2).

The main tool which we use in order to prove the main result is a theorem of Mas-Colell (see [9]), stating that in Aumann's atomless market, each non-competitive allocation in the market can be blocked via a subeconomy by a competitive allocation of the subeconomy. As observed by Greenberg, this theorem can actually be stated as the "external optimistic stability of the core mapping in pure exchange markets" (see [5 and 6]). The fact that for our exact atomless glove-market games, an allocation which is blocked by an allocation in the core of a subeconomy is also blocked by an allocation in the core of the grand coalition is an easy consequence of the characterization in [4] of the core of such atomless games, and the fact that the measures are probability measures. The result follows since in our case, the market as a non-side-payments game is actually a cooperative game with side payments, and the different results are all combined so as to yield our main theorem.

Our main result is used to study stable sets and the core of games which have a decreasing returns to scale property (see Theorems 3.1 and 3.2 and Corollary 3.3). Such games arise in various economic applications (see for example Chapter VI in [3]).

The paper is organized as follows. In Section 2 we state and prove our main result, namely that the core of a glove-market game, which is defined as the minimum of a finite number of non-atomic probability measures, is a stable set. We also give an example which shows that this result cannot be extended to glove-market games with a finite set of players. In Section 3 we use our main result in order to study the core and stable sets of large games which have a decreasing returns to scale property. In Section 4 we study exact glove-market games. In particular we show that if the glove-market game consists of mutually singular non-atomic measures, then its core is a stable set iff the game is exact.

2. Stability of the Core in Non-atomic Glove-Market Games

In this section we state and prove our main result.

Let (T, Σ) be a measurable space. The members of T are called *players*, the members of Σ, *coalitions*. A *coalitional game*, or simply a *game*, is a real valued function v on Σ with $v(\emptyset)=0$. The *core* of a game v on (T, Σ) is the set of all measures λ on (T, Σ) such that $\lambda(S) \geqslant v(S)$ for each coalition S and $\lambda(T)=v(T)$. The core of v will be denoted by Core(v).

We assume that a fixed non-atomic probability measure μ is given on Σ (by a measure we mean in this paper a non-negative σ-additive real valued function which is not identically zero). We interpret μ as a *population measure*, that is, if S is a coalition, then $\mu(S)$ is the proportion of the total population which is contained in S. A *carrier* of a game v on (T, Σ) is a coalition S_0 such that $v(S)=v(S \cap S_0)$ for each $S \in \Sigma$. A game v is *absolutely continuous* with respect to μ if every carrier of μ is also a carrier of v. Let v be a game on (T, Σ) which is absolutely continuous with respect to μ. Denote by D the set of all measures ξ on (T, Σ) which are absolutely continuous with respect to μ and satisfy $\xi(T)=v(T)$. We define a binary relation $\succ$ on D by $\xi \succ \eta$ if and only if there exists a coalition S such that $\mu(S)>0$, $\xi(S) \leqslant v(S)$ and $\xi(S')>\eta(S')$ for each coalition $S' \subset S$ with $\mu(S')>0$. A set $V \subset D$ is a *von Neumann–Morgenstern stable set* (or simply a stable set) of the game v if

(1) V is *internally stable*, i.e., if $\eta \in V$ then there is no $\xi \in V$ such that $\xi \succ \eta$.

(2) V is *externally stable*, i.e., if $\eta \in D \backslash V$ there is $\xi \in V$ such that $\xi \succ \eta$.

If $\mu_1, ..., \mu_n$ are measures on (T, Σ), the game $v(S)=\min(\mu_1(S), ..., \mu_n(S))$ is often referred to as a *glove-market game*.

We are now ready to state our main result.

Main Theorem. *Let $\mu_1, ..., \mu_n$ be non-atomic probability measures on (T, Σ) which are absolutely continuous with respect to μ. For each $S \in \Sigma$ let $v(S)=\min(\mu_1(S), ..., \mu_n(S))$. Then the core of the game v is the unique von Neumann–Morgenstern stable set.*

For the proof of the main result we need some preparation.

Let $\mu_1, ..., \mu_n$ be non-atomic probability measures on (T, Σ) which are absolutely continuous with respect to μ. Consider the game $\bar{v}$ on (T, Σ) which is given by $\bar{v}(S)=\min(\mu_1(S), ..., \mu_n(S))+\sum_{i=1}^{n} \mu_i(S)$ for each $S \in \Sigma$. We associate with $\bar{v}$ a pure exchange economy E as follows :

Let $f_1, ..., f_n$ be the Radon–Nikodym derivatives of $\mu_1, ..., \mu_n$ with respect to μ. That is, $\mu_i(S)=\int_S f_i \, d\mu$ for each $S \in \Sigma$. Each player $t \in T$ is considered to have the initial bundle $\omega(t)=(f_1(t), ..., f_n(t)) \in \mathbf{R}_{+}^{n}$. The utility function $u_t = u$ of $t \in T$ is defined on $\mathbf{R}^n_+$ and is given by $u(x_1, ..., x_n) = \min(x_1, ..., x_n) + \sum_{i=1}^n x_i$. Let $S \in \Sigma$ with $\mu(S) > 0$. An *S-allocation* is a measurable function $x: S \to \mathbf{R}^n_+$ such that $\int_S x \, d\mu = \int_S \omega \, d\mu$. Let x be an S-allocation and $S' \subset S$ be a coalition. We say that x is blocked by S' if $\mu(S') > 0$ and there is an S'-allocation y such that $u(y(t)) > u(x(t))$ for almost all $t \in S'$. The core of the subeconomy E_S is the set of all S-allocations which cannot be blocked by a coalition $S' \subset S$ and is denoted by $\text{Core}(E_S)$. If S is a coalition we denote by $\text{Core}(\bar{v}_S)$ the core of the restriction of the game $\bar{v}$ to S.

We also need the following result of Billera and Raanan (see [4], Corollary 2.7).

PROPOSITION 2.1. *Let $\mu_1, ..., \mu_n$ be non-atomic measures on (T, Σ). For each $S \in \Sigma$ let $v(S) = \min(\mu_1(S), ..., \mu_n(S))$. Then* Core$(v)$ *coincides with the convex hull of the set $\{\mu_i \mid i = 1, ..., n \text{ and } \mu_i(T) = v(T)\}$.*

We are now ready to prove our main result.

Proof of the main result. It suffices to prove that the core is externally stable. Let ξ be a non-atomic probability measure on (T, Σ) which is absolutely continuous with respect to μ such that $\xi \notin \text{Core}(v)$. Let f be the Radon–Nikodym derivative of ξ with respect to μ. Define $\bar{f}(t) = f(t) + \sum_{i=1}^n f_i(t)$ and $\bar{x}: T \to \mathbf{R}^n_+$ by $\bar{x}(t) = 1/(n+1)(\bar{f}(t), ..., \bar{f}(t))$. Then $\int \bar{x} \, d\mu = (1, 1, ..., 1) = \int \omega \, d\mu$ and $\bar{f}(t) = u(\bar{x}(t))$. Let $\bar{\xi} = \xi + \sum_{i=1}^n \mu_i$. Since $\xi \notin \text{Core}(v)$, $\bar{\xi} \notin \text{Core}(\bar{v})$. We will show that $\bar{x} \notin \text{Core}(E_T)$. As $\bar{\xi} \notin \text{Core}(\bar{v})$, there is a coalition S such that $\bar{\xi}(S) < \bar{v}(S)$. We may assume that $\bar{f}(t) > 0$ for all $t \in S$. We show that $\bar{x}$ can be blocked by S. For each $t \in S$ let $\alpha(t) = \bar{f}(t)/\int_S \bar{f} \, d\mu$ and $y(t) = \alpha(t) \int_S \omega \, d\mu$. Then $\int_S y \, d\mu = \int_S \omega \, d\mu$ and thus y is an S-allocation. Now $\int_S \bar{f} \, d\mu = \bar{\xi}(S) < \bar{v}(S)$. Therefore for all $t \in S$

$$u(y(t)) = \alpha(t) \, u\left(\int_S \omega \, d\mu\right) = \alpha(t) \, \bar{v}(S) > \alpha(t) \int_S \bar{f} \, d\mu = \bar{f}(t) = u(\bar{x}(t)).$$

Thus $\bar{x}$ is blocked by S and therefore $\bar{x} \notin \text{Core}(E_T)$. By Aumann's equivalence theorem (see [1]) and Proposition 7.3.2 in [9] there are a coalition $C \in \Sigma$ with $\mu(C) > 0$ and a C-allocation z in $\text{Core}(E_C)$ such that $u(z(t)) > u(\bar{x}(t))$ for all $t \in C$. For each coalition B such that $B \subset C$ let $\bar{\eta}(B) = \int_B u(z(t)) \, d\mu$. If $B \subset C$ is a coalition such that $\mu(B) > 0$, then $\bar{\eta}(B) > \bar{\xi}(B)$. We will show that $\bar{\eta} \in \text{Core}(\bar{v}_C)$. By Jensen's inequality we have $\bar{\eta}(C) = \int_C u(z(t)) \, d\mu \leqslant u(\int_C z \, d\mu) = \bar{v}(C)$. Therefore it is sufficient to show that for each coalition $B \subset C$ such that $\mu(B) > 0$ we have $\bar{\eta}(B) \geqslant \bar{v}(B)$. Assume, on the contrary, that there is a coalition $B \subset C$ with $\mu(B) > 0$ and $\int_B u(z(t)) \, d\mu < u(\int_B \omega \, d\mu)$. Let $\bar{g}(t) = u(z(t))$ and $\beta(t) = \bar{g}(t)/\int_B \bar{g} \, d\mu$ for each

$t \in B$. Define $w(t) = \beta(t) \int_B \omega \, d\mu$ for $t \in B$. Then $\int_B w \, d\mu = \int_B \omega \, d\mu$ and for all $t \in B$

$$u(w(t)) = \beta(t) \, u\left(\int_B \omega \, d\mu\right) = \beta(t) \, \bar{v}(B) > \beta(t) \int_B \bar{g} \, d\mu = \bar{g}(t) = u(z(t)).$$

Thus z is blocked by B, which contradicts the fact that $z \in \mathrm{Core}(E_C)$.

Now for each coalition $B \subset C$ define $\eta(B) = \bar{\eta}(B) - \sum_{i=1}^n \mu_i(B)$. Then $\eta \in \mathrm{Core}(v_C)$ and for each coalition $B \subset C$ such that $\mu(B) > 0$ we have $\eta(B) > \xi(B)$. By Proposition 2.1 there exist non-negative numbers $a_1, ..., a_n$ such that $\sum_{i=1}^n a_i = 1$, $a_i = 0$ for each i with $\mu_i(C) > v(C)$, and $\eta(B) = \sum_{i=1}^n a_i \mu_i(B)$ for each coalition $B \subset C$. Define a measure τ on (T, Σ) by $\tau(A) = \sum_{i=1}^n a_i \mu_i(A)$ for each $A \in \Sigma$. Then τ is a non-atomic probability measure on Σ which belongs to $\mathrm{Core}(v)$. Moreover, $\tau(C) = v(C)$ and $\tau(B) > \xi(B)$ for each coalition $B \subset C$, with $\mu(B) > 0$, and therefore $\tau \succ \xi$. Thus we have shown that $\mathrm{Core}(v)$ is a von Neumann–Morgenstern solution of v.

Remark. Our main result does not hold without the assumption that the measures $\mu_1, \ldots, \mu_n$ have the same mass on T. Indeed, consider the measurable space $([0, 3], \mathbf{B})$, where $\mathbf{B}$ is the σ-field of Borel subsets of $[0, 3]$ and let λ be the Lebesgue measure. Define two measures μ_1, μ_2 on $\mathbf{B}$ by

$$\mu_1(S) = \lambda(S \cap [0, 1]), \qquad \mu_2(S) = \lambda(S \cap [1, 3]).$$

Let $v(S) = \min(\mu_1(S), \mu_2(S))$ for each $S \in \mathbf{B}$. Since $\mu_2([0, 3]) > v([0, 3])$, by Proposition 2.1, $\mathrm{Core}(v) = \{\mu_1\}$.

Let $\xi(S) = \frac{1}{2}\mu_2(S)$ for each $S \in \mathbf{B}$. Assume that the population measure on [0,3] is $\frac{1}{3}\lambda$. Now if $\mu_1(S') > \xi(S')$ for each coalition $S' \subset S$ with $\lambda(S') > 0$, then $\mu_2(S) = 0$. Therefore, $v(S) = 0$, and thus $\mu_1(S) > v(S)$, which implies that $\mu_1 \succ \xi$ does not hold. Thus $\mathrm{Core}(v)$ is not externally stable.

The following example shows that our main result cannot be extended to glove-market games with a finite set of players (for the relation between stable sets of finite markets and those of large markets the reader is referred to [7]).

EXAMPLE 2.2. Let $N = \{1, 2, 3, 4, 5\}$. Define two measures μ_1 and μ_2 on 2^N, the set of all subsets of N, by

$$\mu_1(\{1\}) = \mu_2(\{3\}) = \mu_2(\{4\}) = \mu_2(\{5\}) = \tfrac{1}{3}$$

$$\mu_1(\{2\}) = \tfrac{2}{3}$$

$$\mu_1(\{3\}) = \mu_1(\{4\}) = \mu_1(\{5\}) = \mu_2(\{1\}) = \mu_2(\{2\}) = 0.$$

Let $v(S)=\min(\mu_1(S), \mu_2(S))$ for each $S \in 2^N$. If $(x_1, \ldots, x_5) \in \text{Core}(v)$, then $x_1+x_i \geqslant \frac{1}{3}$ for each $i \in \{3, 4, 5\}$ and $x_2+x_j+x_k \geqslant \frac{2}{3}$ for each $\{j, k\} \subset \{3, 4, 5\}, j \neq k$. Therefore $x_1+x_i=\frac{1}{3}$ and $x_2=2x_1$ for each $i \in \{3, 4, 5\}$, and thus $\text{Core}(v)=\{(a, 2a, \frac{1}{3}-a, \frac{1}{3}-a, \frac{1}{3}-a) \mid 0 \leqslant a \leqslant \frac{1}{3})\}$. Let $y=(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0)$. Then $y \notin \text{Core}(v)$, and it is easy to see that there is no $x \in \text{Core}(v)$ such that $x \succ y$.

Finally we note that our main theorem implies that the Mas-Colell bargaining set (see [10]) of a glove-market game which consists of a finite number of non-atomic probability measures is a stable set. This follows directly from Mas-Colell's theorem [10] asserting that in Aumann's atomless market the bargaining set coincides with the competitive allocations (and hence is equal to the core).

3. Core and Stable Sets of Games with Decreasing Returns to Scale

In this section we use our main theorem to derive some results about the core and von Neumann–Morgenstern stable sets of games with decreasing returns to scale. Such games arise in economic applications.

If $(\mu_1, \ldots, \mu_n)$ is a vector of measures we denote by $R(\mu_1, \ldots, \mu_n)$ the range of $(\mu_1, \ldots, \mu_n)$. Also, if A is a set of measures, we denote by co A the convex hull of A.

Theorem 3.1. *Let $\mu_1, \ldots, \mu_n$ be measures which are absolutely continuous with respect to the population measure μ. Assume that $f: R(\mu_1, \ldots, \mu_n) \to \mathbf{R}_+$ is a function which satisfies $f(0)=0$ and*

(1) f is non-decreasing, i.e., if $x, y \in R(\mu_1, \ldots, \mu_n)$ and $x \geqslant y$, then $f(x) \geqslant f(y)$.

(2) f has decreasing returns to scale along the diagonal, i.e., for each $0 \leqslant \alpha \leqslant 1$ we have $f(\alpha\mu_1(T), \ldots, \alpha\mu_n(T)) \geqslant \alpha f(\mu_1(T), \ldots, \mu_n(T))$.

For each $S \in \Sigma$ let $v(S)=f(\mu_1(S), \ldots, \mu_n(S))$ and $\theta_i(S)=\mu_i(S)/\mu_i(T)$, $i=1, \ldots, n$. Then the set $v(T)\,\text{co}\{\theta_i\}_{i=1}^n$ is externally stable in v and therefore it includes the core of v.

Proof. By (1) and (2), letting $m(S)=\min(\theta_1(S), \ldots, \theta_n(S))$, we have for each $S \in \Sigma$,

$$\begin{aligned} v(S) &= f(\mu_1(S), \ldots, \mu_n(S)) = f(\mu_1(T)\,\theta_1(S), \ldots, \mu_n(T)\,\theta_n(S)) \\ &\geqslant f(m(S)\,\mu_1(T), \ldots, m(S)\,\mu_n(T)) \geqslant m(S)\,v(T). \end{aligned}$$

Since θ_i, $i=1, \ldots, n$, are probability measures, by our main theorem, $v(T)\,\text{co}\{\theta_i\}_{i=1}^n$ is a stable set in the game $\bar{v}(S)=v(T)\,m(S)$. As $v \geqslant \bar{v}$, we obtain that $v(T)\,\mathrm{co}\{\theta_i\}_{i=1}^n$ is externally stable in the game v and therefore it includes Core(v).

Remark. Assume $v(S)=f(\mu_1(S), \ldots, \mu_n(S))$, where $\mu_1, \ldots, \mu_n$ are as in Theorem 3.1 and f is concave and non-decreasing with $f(0)=0$. Then f satisfies (2) in Theorem 3.1 and therefore $v(T)\,\mathrm{co}\{\theta_i\}$ is also externally stable for such a game. An example which is of special interest in economics is the game $v(S)=\prod_{i=1}^n \mu_i^{\alpha_i}(S)$, where $\alpha_i \geqslant 0$ for each $i=1, \ldots, n$ and $\sum_{i=1}^n \alpha_i \leqslant 1$.

THEOREM 3.2. *Assume that $\mu_1, \ldots, \mu_n$ and the function f satisfy the assumptions of Theorem* 3.1, *and in addition $\mu_1, \ldots, \mu_n$ are mutually singular and $f(x)=0$ for each x in $\partial\mathbf{R}_+^n$, the boundary of $\mathbf{R}_+^n$. For each $S\in\Sigma$ let $v(S)=f(\mu_1(S), \ldots, \mu_n(S))$. Then $v(T)\,\mathrm{co}\{\theta_i\}_{i=1}^n$ is a von Neumann–Morgenstern stable set for the game v (θ_i are defined as in Theorem* 3.1).

Proof. By Theorem 3.1 it is sufficient to show that $v(T)\,\mathrm{co}\{\theta_i\}_{i=1}^n$ is internally stable in the game v. Otherwise there exist $\alpha_i, \beta_i \geqslant 0$, $i=1, \ldots, n$, $\sum_{i=1}^n \alpha_i=1$, $\sum_{i=1}^n \beta_i=1$, and a coalition S with $\mu(S)>0$ such that $\sum_{i=1}^n \alpha_i\theta_i(S')>\sum_{i=1}^n \beta_i\theta_i(S')$ for each coalition $S'\subset S$ with $\mu(S')>0$. Since $v(S)>0$, $(\mu_1(S), \ldots, \mu_n(S))\notin\partial\mathbf{R}_+^n$, and thus $\mu_i(S)>0$ for each $1\leqslant i\leqslant n$, which implies that $\theta_i(S)>0$ for each $1\leqslant i\leqslant n$. Since $\theta_1, \ldots, \theta_n$ are mutually singular, there is a coalition $S_0\subset S$ such that $\theta_1(S_0)=\cdots=\theta_n(S_0)>0$. Therefore $\mu(S_0)>0$ and $\sum_{i=1}^n \alpha_i\theta_i(S_0)=\sum_{i=1}^n \beta_i\theta_i(S_0)$, which is a contradiction.

It is well known that if λ is a non-atomic measure on (T, Σ) and f: $R(\lambda)\to\mathbf{R}$ is a function which has a bounded variation on $[0, \lambda(T)]$ and satisfies $f(0)=0$, then the Aumann–Shapley value of the game $v=f\circ\lambda$ is given by $\varphi(v)=(v(T)/\lambda(T))\lambda$ (see [3], Proposition 6.1). By Theorem 3.2 we obtain

COROLLARY 3.3. *Let λ be a measure on (T, Σ) which is absolutely continuous with respect to the population measure μ. Let f: $R(\lambda)\to\mathbf{R}$ be a non-decreasing function on $[0, \lambda(T)]$ which satisfies $f(0)=0$ and $f(\alpha\lambda(T))\geqslant\alpha f(\lambda(T))$ for each $0\leqslant\alpha\leqslant 1$. Then the Aumann–Shapley value of the game $v=f\circ\lambda$ is a von Neumann–Morgenstern stable set.*

The assumption that the function f in Theorem 3.2 vanishes on $\partial\mathbf{R}_+^n$ cannot be removed as we can see in the following example.

EXAMPLE 3.4. Consider the measurable space $([0, 2], \mathbf{B})$ where $\mathbf{B}$ is the σ-field of Borel subsets of $[0, 2]$. The population measure will be $\frac{1}{2}\lambda$, where λ is the Lebesgue measure. Define $\mu_1(S)=\lambda(S\cap[0, 1])$ and $\mu_2(S)=\lambda(S\cap[1, 2])$ for each $S\in\mathbf{B}$. Let $f(x, y)=x+y$. Then f satisfies the assumptions of Theorem 3.1. Let $v = f \circ (\mu_1, \mu_2)$. Then $v = \lambda$ and so $\{\lambda\}$ is the unique stable set for v. Thus $2\,\mathrm{co}\{\mu_1, \mu_2\}$ is not a stable set.

The next example shows that the assumption in Theorem 3.2 that $\{\mu_i\}_{i=1}^n$ are mutually singular cannot be dropped.

EXAMPLE 3.5. Consider the same measurable space as in Example 3.4. For each $S \in \mathbf{B}$ let $\mu_1(S) = \frac{1}{3}\lambda(S \cap [0, 1]) + \frac{2}{3}\lambda(S \cap [1, 2])$ and $\mu_2(S) = \frac{2}{3}\lambda(S \cap [0, 1]) + \frac{1}{3}\lambda(S \cap [1, 2])$. Define $f(x, y) = \sqrt{xy}$ and let $v = f \circ (\mu_1, \mu_2)$. Then it is easy to see that $\frac{1}{4}\mu_1 + \frac{3}{4}\mu_2$ dominates μ_2 via the coalition $[1, 2]$ in the game v, and thus $\mathrm{co}\{\mu_1, \mu_2\}$ is not internally stable.

Theorem 3.2 provides sufficient conditions for the existence of von Neumann–Morgenstern stable sets in games which have a decreasing returns to scale property. Moreover, it describes explicitly a stable set for the game. A natural question which can be raised is about the uniqueness of this stable set. The following example shows that it may not be unique.

EXAMPLE 3.6. Consider the measurable space $([0, 1], \mathbf{B})$, where $\mathbf{B}$ is the σ-field of Borel subsets of $[0, 1]$. The population measure will be the Lebesgue measure λ. Consider the game $v(S) = \sqrt{\lambda(S)}$. By Theorem 3.2, $\{\lambda\}$ is a stable set for v. Let $\xi(S) = 2\lambda(S \cap [0, \frac{1}{2}])$ for each $S \in \mathbf{B}$. It is not difficult to see that $\{\xi\}$ is also a stable set for v.

4. EXACT GLOVE-MARKET GAMES

A game v on (T, Σ) is *exact* if for each $S \in \Sigma$ there is $\lambda \in \mathrm{Core}(v)$ such that $v(S) = \lambda(S)$. Exact games were introduced in [15] and in [13]. Let $v(S) = \min(\mu_1(S), \ldots, \mu_n(S))$, where $\mu_1, \ldots, \mu_n$ are non-atomic measures on (T, Σ). Without loss of generality we may assume that for each $1 \leqslant i \leqslant n$ there is $S \in \Sigma$ such that $v(S) = \mu_i(S)$ and $v(S) < \mu_j(S)$ for $j \neq i$. We observe that a necessary and sufficient condition for v to be exact is that $\mu_i(T) = v(T)$ for each $1 \leqslant i \leqslant n$. Indeed, if the condition is satisfied then $\mu_i \in \mathrm{Core}(v)$ for each $1 \leqslant i \leqslant n$. Given $S \in \Sigma$, we can find $1 \leqslant i \leqslant n$ such that $v(S) = \mu_i(S)$, so v is exact. Conversely, suppose that v is exact but there is i such that $\mu_i(T) > v(T)$. By our assumption there is $S \in \Sigma$ such that $\mu_i(S) = v(S)$ and $v(S) < \mu_j(S)$ for $j \neq i$. Therefore by Proposition 2.1, there is no $\lambda \in \mathrm{Core}(v)$ such that $\lambda(S) = v(S)$, which contradicts the fact that v is exact. Thus our main theorem implies that if a glove-market game, which is absolutely continuous with respect to μ is exact, then its core is a stable set. A natural question that can be asked is if the converse also holds, i.e., is it true that a non-atomic glove-market game which has a stable core is exact? As we shall see later in general this is not true (see Example 4.3), but we can prove the following.

THEOREM 4.1. *Let $\mu_1, \ldots, \mu_n$ be mutually singular non-atomic measures on (T, Σ) which are absolutely continuous with respect to μ. Then the core of the game $v(S) = \min(\mu_1(S), \ldots, \mu_n(S))$ is a stable set iff the game v is exact.*

Proof. By the above discussion and our main result it is clear that if v is exact its core is a stable set. Assume now that the core of v is a stable set. We show that v is exact. Assume not. Then there exists $1 \leqslant i \leqslant n$ such that $\mu_i(T) > v(T)$. Define μ_i^* by $\mu_i^*(S) = (v(T)/\mu_i(T))\, \mu_i(S)$ for each $S \in \Sigma$. By Proposition 2.1, $\mu_i^* \notin \text{Core}(v)$. Since Core$(v)$ is a stable set, there is $\lambda \in \text{Core}(v)$ such that $\lambda \succ \mu_i^*$. Let S_i be the support of μ_i. By Proposition 2.1, $\lambda(S_i) = 0$. Since $\lambda \succ \mu_i^*$ there is a coalition S such that $\mu(S) > 0$, $\lambda(S) = v(S)$ and $\lambda(S') > \mu_i^*(S')$ for each coalition $S' \subset S$ with $\mu(S') > 0$. Therefore $\mu_i(S) = \mu_i(S \cap S_i) = 0$ and thus $v(S) = 0$, which contradicts the fact that $\lambda(S) > 0$.

COROLLARY 4.2. *Let μ_1, μ_2 be two non-atomic measures on (T, Σ) which are absolutely continuous with respect to μ. Then the core of the game $v(S) = \min(\mu_1(S), \mu_2(S))$ is a stable set iff v is exact.*

Proof. Let f_1 and f_2 be the Radon–Nikodym derivatives of μ_1 and μ_2 with respect to μ, respectively, and let $\lambda(S) = \int_S \min(f_1, f_2)\, d\mu$ for each $S \in \Sigma$. Define $\lambda_i = \mu_i - \lambda$, $i = 1, 2$, and let $\bar{v}(S) = \min(\lambda_1(S), \lambda_2(S))$. Then $\xi \in \text{Core}(v)$ iff $\xi - \lambda \in \text{Core}(\bar{v})$. Since λ_1 and λ_2 are mutually singular, by Theorem 4.1 Core$(\bar{v})$ is a stable set iff $\bar{v}$ is exact. Since Core(v) is a stable set iff Core$(\bar{v})$ is a stable set, and v is exact iff $\bar{v}$ is exact, the result follows.

The following example shows that the assumption that the measures μ_i in Theorem 4.1 are mutually singular is necessary.

EXAMPLE 4.3. Consider the measurable space $([0, 1], \mathbf{B})$, where $\mathbf{B}$ is the set of Borel subsets of $[0, 1]$. The population measure will be the Lebesgue measure λ on $([0, 1], \mathbf{B})$. For each $S \in \mathbf{B}$ let $\mu_1(S) = 2\lambda(S \cap [0, \frac{1}{2}])$ and $\mu_2(S) = 2\lambda(S \cap [\frac{1}{2}, 1])$. Let μ_3 be the measure with density function

$$f_3(x) = \begin{cases} 0 & 0 \leqslant x < \frac{1}{4} \\ 1 & \frac{1}{4} \leqslant x < \frac{5}{8} \\ 2 & \frac{5}{8} \leqslant x \leqslant 1. \end{cases}$$

For each $S \in \mathbf{B}$ let $v(S) = \min(\mu_1(S), \mu_2(S), \mu_3(S))$. Since $\mu_3([0, 1]) = \frac{3}{8}(1 + 2) = \frac{9}{8}$, by Proposition 2.1, Core(v) consists of the convex hull of μ_1 and μ_2. Now the coalition $S = [0, \frac{1}{8}] \cup [\frac{1}{2}, \frac{5}{8}]$ gets $\frac{1}{4}$ in Core(v) and $v(S) = \mu_3(S) = \frac{1}{8}$. Therefore v is not exact. We will show that Core(v) is a stable set. Suppose, for contradiction, that $\mu \in D \backslash \text{Core}(v)$ and there is no

$\xi \in \text{Core}(v)$ such that $\xi \succ \mu$. Let f be the Radon–Nikodym derivative of μ with respect to λ. Let m_1, m_2, m_3, m_4 denote the essential infimum of f (with respect to λ) in the intervals $[0, \frac{1}{4})$, $[\frac{1}{4}, \frac{1}{2})$, $[\frac{1}{2}, \frac{5}{8})$, $[\frac{5}{8}, 1]$ respectively. Then $1 = \mu([0, 1]) = \int f \, d\lambda \geqslant \frac{1}{4}m_1 + \frac{1}{4}m_2 + \frac{1}{8}m_3 + \frac{3}{8}m_4$. If $m_2 + m_4 < 2$ then we can find for some $\varepsilon, \delta > 0$ two subsets of λ measure ε of $[\frac{1}{4}, \frac{1}{2})$ and $[\frac{5}{8}, 1]$, respectively, such that over the first $f < m_2 + \delta$, over the second $f < m_4 + \delta$, and $m_2 + m_4 + 2\delta = 2$. Then the core element giving $m_2 + \delta$ on $[0, \frac{1}{2}]$ and $m_4 + \delta$ on $(\frac{1}{2}, 1]$ dominates μ, contrary to our assumption. Therefore $m_2 + m_4 \geqslant 2$. Similarly $m_2 + m_3 \geqslant 2$ and $m_1 + m_4 \geqslant 2$. Putting $m_i = 1 + e_i$, $i = 1, ..., 4$, we get

$$\begin{aligned} e_2 + e_4 &\geqslant 0 \\ e_2 + e_3 &\geqslant 0 \\ e_1 + e_4 &\geqslant 0 \\ \tfrac{1}{4}e_1 + \tfrac{1}{4}e_2 + \tfrac{1}{8}e_3 + \tfrac{3}{8}e_4 &\leqslant 0. \end{aligned}$$

Using $e_3 \geqslant -e_2$ we get

$$0 \geqslant \tfrac{1}{4}e_1 + \tfrac{1}{4}e_2 - \tfrac{1}{8}e_2 + \tfrac{3}{8}e_4 = \tfrac{1}{4}e_1 + \tfrac{1}{8}e_2 + \tfrac{3}{8}e_4.$$

Now $e_4 \geqslant -e_2, -e_1$, so

$$0 \geqslant \tfrac{1}{4}e_1 + \tfrac{1}{8}e_2 - \tfrac{3}{8}\begin{cases} e_1 \\ e_2 \end{cases} = \begin{cases} \tfrac{1}{8}(e_2 - e_1) \\ \tfrac{1}{4}(e_1 - e_2). \end{cases}$$

The last inequality implies that $e_1 = e_2$ and $e_3 = e_4 = -e_1$. Therefore $m_1 = m_2 = 1 + e_1$ and $m_3 = m_4 = 1 - e_1$ and thus $\int f \, d\lambda \geqslant \frac{1}{4}m_1 + \frac{1}{4}m_2 + \frac{1}{8}m_3 + \frac{3}{8}m_4 = 1$, which implies that $f = m_i$ almost everywhere in the respective intervals, so $\mu \in \text{Core}(v)$, which is a contradiction.

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