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IMPARTIAL NOMINATIONS FOR A PRIZE

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IMPARTIAL NOMINATIONS FOR A PRIZE

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A group of peers must choose one of them to receive a prize; everyone cares only about winning, not about who gets the prize if someone else. An award rule is *impartial* if one's message never influences whether or not one wins the prize. We explore the consequences of impartiality when each agent nominates a single (other) agent for the prize.

On the positive side, we construct impartial nomination rules where both the influence of individual messages and the requirements to win the prize are not very different across agents. Partition the agents in two or more *districts*, each of size at least 3, and call an agent a *local winner* if he is nominated by a majority of members of his own district; the rule selects a local winner with the largest support from nonlocal winners, or a fixed default agent in case there is no local winner.

On the negative side, impartiality implies that ballots cannot be processed anonymously as in plurality voting. Moreover, we cannot simultaneously guarantee that the winner always gets at least one nomination, and that an agent nominated by everyone else always wins.

KEYWORDS: Impartiality, plurality, positive and negative unanimity, monotonicity.

1. INTRODUCTION

1.1. *Impartiality*

THE POSSIBILITY OF AN IMPARTIAL JUDGMENT is a cornerstone of modern theories of justice, from Harsanyi's impartial observer (Harsanyi (1955)), to Rawls' veil of ignorance (Rawls (1971)), and Sen's transpositional objectivity (Sen (2009)). In the more mundane context of committees and elections, impartial evaluations are a desirable but elusive ingredient of group decision making. When individual opinions are aggregated into a collective outcome, an agent may be tempted to corrupt her valuable disinterested opinion (which influences the final decision) to serve her selfish preferences. Thus corrupted, the profile of messages may yield a suboptimal decision. Avoiding such conflicts of interest is a tall order, particularly in the context of evaluation by peers, a central institution in many communities of experts.

Here we study *nomination rules* to award a prize among peers. A group of agents must choose one of them to receive a prize (or undertake a task); each agent is asked to nominate someone (other than self) as the most worthy of the prize (or the best qualified for the task), and the profile of nominations

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determines the final winner. The key assumption is that an agent's selfish preferences bear on whether or not she gets the prize/task,[2] and nothing else, while she has a disinterested opinion about who should get it if not herself.[3]

We call a nomination rule *impartial* if reporting one's disinterested opinion never affects whether or not one wins the prize/task. In the absence of selfish incentives to distort such reports, the outcome aggregates information that is not corrupted by any conflict of interest.

Our assumption on selfish preferences is an extreme restriction to a domain of size 3 (full indifference, or exactly two indifference classes of size 1 and $n-1$). Our modeling approach is that the decision rule should ignore by design these simple binary preferences, so that, for instance, whether or not the outcome of the nomination game is Pareto inferior with respect to selfish preferences (Jack wins but prefers not to, while Jill would like to win) is deemed irrelevant. In the noncooperative nomination game of an impartial rule, every strategy of every player is dominant, so we can ignore incentives and focus on the normative analysis of the rule, pertaining to the way it aggregates the disinterested opinions of the agents.

Our goal is to find "reasonable" impartial nomination rules. The examples discussed in the next subsection illustrate the difficulty of reaching this goal.

1.2. *Examples and Main Axioms*

Given the finite set of agents N, $|N| = n \geq 2$, we write N_{-}^{N} for the set of nomination profiles: $N_{-}^{N} = \{x \in N^N | x_i \in N \setminus \{i\} \text{ for each } i\}$.

DEFINITION 1: A nomination rule is a function $\varphi : N_{-}^{N} \to N$.

We use the following notational conventions. Given a profile of nominations $x \in N_{-}^{N}$, we write $\delta(x) = s$ for the profile of scores at x:

$$s_i = \left|\{j \in N | x_j = i\}\right|.$$

A profile of nominations by all agents except i is written x_{-i}, and the set of all such profiles is $N_{-}^{N\setminus i}$. Given $x_{-i} \in N_{-}^{N\setminus i}$, we can specify i's nomination $x_i \in N \setminus \{i\}$ and obtain $(x_i, x_{-i}) \in N_{-}^{N}$.

Our central axiom is the following:

- *Impartiality*: for all $i \in N$, $x_i, x_i' \in N \setminus \{i\}$, and all $x_{-i} \in N_{-}^{N\setminus i}$,

$$\varphi(x_i, x_{-i}) = i \quad \Leftrightarrow \quad \varphi\left(x_i', x_{-i}\right) = i.$$

[2]Some agents may relish to be assigned the task while others view it as a painful chore.

[3]In a more general context, Sen (1977) proposed a taxonomy of social choice problems when agents are endowed with two orderings over outcomes, one representing their honest opinion/views, the other their selfish interest.

The most natural nomination rule is *plurality rule*, defined by $\varphi(x) \in \arg\max_{i\in N}\{s_i\}$ for all x, with some tie-breaking rule. While i's vote x_i does not influence i's score s_i, it does influence other agents' scores and thereby i's relative standing. Hence plurality rule, irrespective of the way ties are broken, is not impartial.[4]

To achieve Impartiality with a rule "close" to plurality, we choose a fixed *default agent* i^* and write $s_i(-j,k)$ for the score of i in $N \setminus \{j,k\}$ (i.e., not counting j and k's votes). Then we define the rule *plurality with default*, Plu^{i^*}, as follows:

$$\text{if for some } i \neq i^*: \quad \{s_i(-j,i^*) > s_j(-i,i^*) \text{ for all } j \neq i\},$$
$$\text{then } \mathrm{Plu}^{i^*}(x) = i, \text{otherwise } \mathrm{Plu}^{i^*}(x) = i^*.$$

This rule is clearly well defined; it is impartial because agent i does not influence either score $s_i(-j,i^*)$ or $s_j(-i,i^*)$, and i^* has no influence at all on the outcome. But it is unpalatable for three reasons:

(1) the message of the default agent i^* is ignored;
(2) the default i^* can win without any support ($s_{i^*} = 0$);[5]
(3) additional votes for i^* can turn him from a winner to a loser.[6]

We introduce three axioms that rule out these three undesirable features, respectively:

- *No Dummy*: for all $i \in N$, there exist $x_i, x'_i \in N \setminus \{i\}$ and $x_{-i} \in N_{-}^{N\setminus i}$ such that

$$\varphi(x_i, x_{-i}) \neq \varphi(x'_i, x_{-i}).$$

- *Negative Unanimity*: for all $x \in N_{-}^{N}$ and all $i \in N$,

$$s_i = 0 \quad \Rightarrow \quad \varphi(x) \neq i.$$

- *Monotonicity*: for all $i, j \in N$, $x_j, x'_j \in N \setminus \{j\}$, and all $x_{-j} \in N_{-}^{N\setminus j}$,

$$\{\varphi(x_j, x_{-j}) = i \text{ and } x'_j = i\} \quad \Rightarrow \quad \varphi(x'_j, x_{-j}) = i.$$

A simple modification of Plu^{i^*} turns it into a monotonic and impartial rule, *majority with default*, denoted Maj^{i^*}. A nondefault agent i wins if and only if he

[4] To check this, consider a profile x, where $s_i = 1$ for all i, and let $\varphi(x) = j$. Then by switching his vote to $k \neq x_j$, agent j makes k the unique plurality winner, in contradiction of Impartiality.

[5] This happens, for instance, when, ignoring i^*'s vote, everyone in $N \setminus \{i^*\}$ gets one vote.

[6] For example, suppose $N = \{1,\dots,5\}$ and $i^* = 5$; if $x_{-5} = (2,1,2,1)$, we have $\mathrm{Plu}^{i^*}(x) = i^*$ because agents 1 and 2 are tied; but for $x'_{-5} = (2,1,5,1)$, we have $\mathrm{Plu}^{i^*}(x') = 1$. It should be clear that we can construct similar examples for any $n \geq 6$.

gets a strict majority of the votes that are counted:

$$\mathrm{Maj}^{i^*}(x) = i \quad \text{for some } i \neq i^* \quad \text{if} \quad s_i(-i^*) \geq \left\lceil \frac{n}{2} \right\rceil,$$

$$\text{otherwise } \mathrm{Maj}^{i^*}(x) = i^*$$

(where $\lceil z \rceil$ is the smallest integer weakly above z). Clearly, majority with default i^* selects i^* more often than plurality with default i^* ($\mathrm{Plu}^{i^*}(x) = i^* \Rightarrow \mathrm{Maj}^{i^*}(x) = i^*$), so it is even more biased in favor of i^* (and No Dummy and Negative Unanimity still fail).

To achieve simultaneously No Dummy, Negative Unanimity, and Monotonicity, we use a variant of Maj^{i^*}, *majority with default-maker*, denoted Maj_{i^0}. Here the role of the special agent i^0 is to choose the default agent $x_{i^0} = j$ who wins if no one in $N \setminus \{i^0\}$ garners an absolute majority:

$$\mathrm{Maj}_{i^0}(x) = i \quad \text{for some } i \neq i^0, j \quad \text{if} \quad s_i(-j) \geq \left\lceil \frac{n}{2} \right\rceil,$$

$$\text{otherwise } \mathrm{Maj}_{i^0}(x) = j.$$

Agent i^0 is clearly not a dummy, and everyone else can be pivotal to create an absolute majority winner. The default agent j is supported by i^0, implying Negative Unanimity, and Monotonicity is equally clear. But Impartiality forbids to ever make i^0 a winner, no matter how much support he receives: i^0's message affects $s_{i^0}(-j)$, because it neutralizes a vote in $N \setminus \{i^0\}$. Therefore, the range of Maj_{i^0} is $N \setminus \{i^0\}$, and this rule violates the following axiom:

- *No Exclusion*: for all $i \in N$, there exists $x \in N_-^N$ such that $\varphi(x) = i$.

In the same spirit, we consider also the following compelling axiom:

- *Positive Unanimity*: for all $x \in N_-^N$ and all $i \in N$,

$$s_i = n - 1 \quad \Rightarrow \quad \varphi(x) = i.$$

Positive Unanimity strengthens No Exclusion, but for a monotonic impartial rule the two properties are equivalent.

1.3. *Outline*

In Section 2, we construct a family of monotonic impartial nomination rules excluding no one, dubbed the *partition methods.* In those, both the influence of individual messages and the requirements to win the prize are much more evenly spread across agents than in plurality with default, majority with default, or majority with default-maker. In particular, every agent i *influences* every other agent j, in the sense that, at some profile of nominations, j wins or loses depending on i's vote.[7] See Theorem 1. Theorem 2, also in Section 2, proposes a family of more complex rules in the same vein, where each participant can be *pivotal* over any pair of other agents: for any triple i, j, k, at some profile of messages, i's ballot can change j's win to k's win.

Section 3 uncovers two systematic limitations of impartial nomination rules. First, individual ballots cannot be processed in an anonymous urn as in plurality voting (Theorem 3). We regard our second impossibility result (Theorem 4) as more severe; it is also much harder to prove. No impartial nomination rule simultaneously guarantees that the winner always gets at least one nomination (Negative Unanimity), and that an agent nominated by everyone else always wins (Positive Unanimity). For instance, majority with default meets Positive, but not Negative Unanimity, while majority with default-maker does the opposite.

The theorem implies that any monotonic, nonexcluding, and impartial nomination rule must sometimes give the prize to an agent who is not nominated by anybody (Corollary to Theorem 4). This explains the inevitable occurrence of winning by default in all our constructions, including the partition methods.

The difficult proof of Theorem 4 involves a symmetrization argument in the more general model of *randomized* nomination rules; it is the subject of Section 4.

Section 5 briefly discusses impartial rules to award the prize where each agent sends an abstract message, rather than a nomination. Some concluding comments are gathered in Section 6.

1.4. *Related Literature*

This paper is the first, to our knowledge, to explore impartiality in the setting of nomination rules. There do exist earlier studies of impartiality in different but related models. We comment on three such works here.

1. The concept of impartial decision making appears first in de Clippel, Moulin, and Tideman (2008), applied to the division of a cash surplus within a group of partners. Each partner cares selfishly about his share, not about the distribution among others of the money he does not get. Partners report their subjective opinion about the *relative* contributions of the *other* partners to the surplus; Impartiality requires that one's report has no impact on one's final share. With four or more partners, there exist symmetric impartial rules with the additional property that if there is a *consensual division* compatible with all reports, it is implemented. The prize award problem is clearly the indivisible counterpart of the "division of a dollar" problem; however, the rules there are not similar to those in Section 2 below.

[7]This implies a bossy decision method: i can change j's welfare without affecting his own. Bossiness is a desirable feature in our context, contrary to the standard view in mechanism design literature, going back to Satterthwaite and Sonnenschein (1981).

2. Alon, Fischer, Procaccia, and Tennenholtz (2011) studied a model of "selection from the selectors" conceptually related to the present paper. Each agent can "approve" of an arbitrary subset of other agents, and the rule must choose a fixed number k of winners. Noting that approval voting, that is, selecting k agents with the highest approval scores, is not impartial, Alon et al. sought impartial rules that approximate it (guarantee that the total approval scores of the k winners be at least a fixed fraction of the optimal one). There is an essential difference in approach between this work and ours: rather than fixing a target rule and trying to approximate it subject to impartiality, we explore impartial rules axiomatically. In spite of this difference, Alon et al.'s impossibility theorem for impartial approximation of approval voting (their Theorem 3.1) may be rephrased axiomatically: No rule for selecting k agents in the model of Alon et al. (2011) satisfies the analogs of Impartiality and Negative Unanimity. This statement is similar to our Theorem 4, but the two results are not logically comparable. On the one hand, Alon et al.'s impossibility applies to any number k of winners; ours—only to $k = 1$. On the other hand, our impossibility uses a more restricted domain of messages, from the agents (just single nominations, not arbitrary subsets of other agents), which renders it stronger and more difficult. It is stronger because, on our domain of messages, Impartiality and Negative Unanimity *are* compatible, and impossibility is reached only when Positive Unanimity is also required. It is more difficult because Alon et al.'s proof of Theorem 3.1 depends crucially on abstentions—messages approving no one.[8] On the positive side, we note that the nomination rules we describe in Section 2 (as well as the examples in Section 1.2, except majority with default-maker) work just as well when we allow abstentions: the definition is identical, and the only difference is that the sum of all scores is no longer a constant.

Finally, we remark that Alon et al. also considered randomized selection rules, and obtained a positive result for them; we use randomized nomination rules in the proof of our negative result—Theorem 4.

3. It is possible to interpret impartial nomination rules in the context of the sizeable literature[9] on strategyproof allocation of private goods in Arrow–Debreu economies, going back to Hurwicz's seminal work. Fix a bundle of resources ω to be allocated between our n agents, each one endowed with a (private) strictly monotonic preference R_i chosen in some preference domain

[8]To see how abstentions facilitate the proof, consider, in the case $k = 1$, the profile where all agents abstain, and say that i is the winner. If i changes his message and nominates j, Impartiality requires that i still wins. But in the new profile, some agent (j) has a positive score, yet an agent with score zero (i) wins, contradicting Negative Unanimity. This trivial observation covers the common ground of Alon et al.'s Theorem 3.1 and our Theorem 4. Their result extends it nontrivially by considering any number k of winners; ours uncovers a deeper impossibility by precluding abstentions.

[9]See, in particular, Dasgupta, Hammond, and Maskin (1979), Satterthwaite and Sonnenschein (1981), Zhou (1991), Serizawa (2002), Serizawa and Weymark (2003).

$\mathcal{R}_i$. Pick a nomination rule φ and attach to each preference $R_i \in \mathcal{R}_i$ an arbitrary message x_i in $N \setminus \{i\}$, so that the mapping from preferences to messages is onto. At a preference profile R, give *all* the resources ω to the agent $\varphi(x)$: this (direct revelation) mechanism is strategyproof[10] *if and only if* φ is impartial. This is the viewpoint of Kato and Ohseto (2002, 2004), except that they worked in a more general model with abstract messages that convey no opinion about who should get the prize. The impartial mechanisms they constructed meet No Dummy and No Exclusion as well, thus disproving an earlier conjecture in Zhou (1991): with four or more agents, strategyproofness and efficiency do not imply, as Zhou conjectured, that one of them is never allocated anything. Proposition 2 in Section 5 presents (a variant of) their interesting mechanisms.

2. MUTUAL INFLUENCE AND THE PARTITION METHODS

In the two monotonic and impartial rules Maj^{i^*} and Maj_{i^0}, the role of the designated agent is very special. In the former, i^* wins the prize much more often than any other agent, but he is an entirely passive dummy. In the latter rule, i^0 is very influential because he chooses the default, but he can never win.

In this section, we construct two large families of monotonic and impartial nomination rules where the treatment of the "special agent" is much less extreme on both counts.[11] They satisfy both No Dummy and No Exclusion, and in fact stronger properties that we introduce below. These properties assure in a limited sense a fair distribution of the ability to influence the final outcome.

Given a nomination rule φ and three distinct agents i, j, j', we say that agent i is *pivotal* for the pair j, j' if, for some profile $x_{-i} \in N_{-}^{N\setminus i}$, there exist $x_i, x_i' \in N \setminus \{i\}$ such that

$$\varphi(x_i, x_{-i}) = j, \quad \varphi\big(x_i', x_{-i}\big) = j'.$$

For two distinct agents i, j, we say that *i influences j* if there exists j' so that i is pivotal for j, j'. These notions lead to the following properties of a nomination rule φ, taking it closer to a symmetric distribution of the decision power:

- *Full Pivots*: for all distinct $i, j, j' \in N$, agent i is pivotal for j, j'.
- *Full Influence*: for all distinct $i, j \in N$, agent i influences j.

Clearly, for $n \geq 4$, Full Pivots is stronger than Full Influence, which in turn is a common strengthening of No Exclusion and No Dummy.

The rest of this section is devoted to the construction of monotonic impartial nomination rules satisfying Full Influence (Theorem 1) or Full Pivots (Theorem 2). We give first an intuitive description of those rules, called *partition*

[10] Note, however, that strategyproofness cannot be interpreted in the usual way to mean the elicitation of sincere preferences, for these preferences are entirely clear: everyone wants ω.

[11] Note that fully equal treatment of all agents is impossible for nomination rules, regardless of impartiality. Requiring that the winner determination should not depend on the names of the agents leads to an immediate contradiction for a cyclic profile of nominations.

methods, as constituency-based single-winner voting systems. The agents are partitioned into districts (or constituencies). Every agent may nominate any other agent, inside or outside his district. The winner is determined by a two-step procedure. In the first step, for each district, we check if there is a member who was nominated by an absolute majority[12] of that district; such an agent is designated as a *local winner*. In the second step, the various local winners become the candidates and all other agents become the voters in a plurality vote (using the original nominations) that determines the final winner. Because there may be no local winner anywhere, we need to pre-assign one agent as the default agent who wins in that case, and to preserve impartiality the default agent's vote is ignored in the first step.[13]

The grouping of candidates in districts and two-tier structure of the decision process is often natural: think of awarding a scientific prize among scholars from different fields of research. Our rules check first in each field for a strong candidate, gathering on her name an absolute majority of that field; then almost everyone is involved in the selection of the overall winner among these strong field specialists.

In the formal definition, we use the notation $s_i(B)$ to mean the number of nominations agent i gets from the agents in B.

DEFINITION 2—Partition Methods: Assume $n \geq 7$ and fix a partition $N = \bigcup_{k=1}^{K} N_k$ with $K \geq 2, |N_1| \geq 4, |N_k| \geq 3$ for $k = 2, \dots, K$. We refer to $N_1, \dots, N_K$ as districts $1, \dots, K$, respectively. Assign a particular member of district 1, $i^* \in N_1$, as the default agent. Given a profile of nominations $x \in N_-^N$, the winner $\varphi(x)$ is determined in two steps.

Step 1. Set the absolute majority quota q_k in district k to be

$$q_k = \left\lceil \frac{|N_k| + 1}{2} \right\rceil, \quad k = 2, \dots, K,$$

$$q_1 = \left\lceil \frac{|N_1|}{2} \right\rceil.$$

Call agent $i \in N_k$ a *local winner* if $k \in \{2, \dots, K\}$ and $s_i(N_k) \geq q_k$, or if $k = 1$ and $s_i(N_1 \setminus \{i^*\}) \geq q_1$.

[12]By using absolute majority (rather than plurality), we make sure that an agent cannot affect his own status as a local winner. Certain other variants would work as well, such as some special majority or relative quota (e.g., an agent must get at least two more nominations from his district than any other district member, not counting his own vote). To keep the presentation simple, we stick to absolute majority.

[13]Note that the default agent is still a member of one of the districts, and may win in the same way as other agents, as well as by default. A variant of the method, leaving the default agent outside the partition into districts, would also work; but it would single out that agent in more ways than in the version presented here.

Let X be the set of local winners in the various districts. If $X = \emptyset$, then $\varphi(x) = i^*$. If $|X| = 1$, say $X = \{i\}$, then $\varphi(x) = i$.

Step 2. If $|X| \geq 2$, let $Y = \arg\max_{i \in X} s_i(N \setminus X)$. Using a fixed priority ordering of the set of districts to break ties, the winner $\varphi(x)$ is the member of Y coming from a district with the highest priority among members of Y (the priority ordering is arbitrary, except for the following provision: if $K = 2$ and exactly one of the districts has $q_k = 2$, then that district should have priority).[14]

THEOREM 1: *Partition methods satisfy Monotonicity, Impartiality, and Full Influence.*

PROOF: We start with Impartiality. Obviously, there is at most one local winner in each district. Suppose agent i, $i \neq i^*$, wins the prize. By changing his vote, he cannot affect the fact that he is the unique local winner in his district, or alter the set of local winners in other districts, and his victory in the vote among local winners, if such arises, is also independent of his own message. Suppose next i^* wins. If i^* is the local winner in district 1, the above argument applies. If i^* wins by default, that is, there is no local winner anywhere, he cannot move to create a local winner because his vote is ignored in step 1.

To verify Full Influence, it suffices to show that whenever i and j are distinct members of one district, and j' is in another district, agent i is pivotal for the pair j, j'. For this, we prescribe a profile x_{-i} of nominations by the agents in $N \setminus \{i\}$, so that: (a) j' is a local winner (the only one outside i's district), (b) if i nominates j, then j is a local winner and beats j' in step 2, (c) if i nominates j', then either j is not a local winner anymore (so j' wins in step 1) or j remains a local winner but loses to j' in step 2. The fact that this can be arranged is easily checked: care is required only in tipping the scales in step 2, which is facilitated by the special provision on the priority ordering of the districts.

Finally, we check Monotonicity. Suppose that initially agent i wins, and then j changes his vote to i. If $i = i^*$ and he won by default, the change cannot make anyone else a local winner; either it makes him the only local winner or there is still no local winner, and he continues to win either way. If i won by being a local winner, he remains a local winner, and the set of local winners either remains the same or loses a member, say j', from j's district as a result of the change. In the former case, the change clearly cannot hurt i in step 2. In the latter case, the change results in agent j' turning from candidate to voter in step 2. His vote may go to an opponent of i, but i now gets the vote of j in addition to those he got earlier, so he still wins. *Q.E.D.*

[14]This provision is designed to ensure that any agent may prevail over an agent from any other district in situations where they both are local winners. For example, with two districts of size 4 each, and local winners in both, the one in district 2 (having $q_2 = 3$) must be getting at least three nominations, so district 1 (having $q_1 = 2$) should have priority so as to give its local winner a chance to prevail.

Just like in Maj^{i^*}, in a partition method there is a single agent i^* who can win without any support: this happens only if there is no local winner in any district. It may be argued that under certain plausible assumptions, this default case is highly unlikely to happen. If the partition into districts is natural, it is reasonable to assume that agents are more likely to vote for "one of their own." In a given district, the likelihood of there being a local winner is then at least nonnegligible. If we further assume a large number of districts, it becomes highly unlikely that all of them will be deadlocked. We can then expect to resort to the default option only very rarely.

Partition methods meet all the requirements discussed so far, with two exceptions. They violate Negative Unanimity, as the default agent can win with no support. They do not meet Full Pivots, because an agent is not pivotal for a pair of agents in a different district than his own. To see this, say i is in district 1 and $j \neq j'$ are in district 2. If, for a certain profile of nominations, j wins, then he must be the local winner in district 2. No matter how i changes his vote, j remains a local winner in district 2, so j' cannot be a local winner there, and hence cannot win as a result of the change. In addition, it may be the case that an agent is not pivotal for a pair of agents in his own district (this depends on parity).

The failure of Full Pivots for partition methods can be fixed, at the cost of a more complicated definition, as follows.

DEFINITION 3—Cross-Partition Methods: Assume $n \geq 13$, and partition the agents into $K \geq 3$ districts N_k of nearly equal size[15] (at least 4 each), and the singleton $\{i^*\}$. For each $k = 1, \ldots, K$, let Π_k be a partition of N_k into two components of nearly equal size, and let Λ_k be another such partition, so that the two are orthogonal (in the sense that each component of one intersects each component of the other). Given a profile of nominations x, the winner $\varphi(x)$ is determined in two steps.

Step 1. Call agent $i \in N_k$ an *outer hero* if he gets the votes of an entire component of Π_l for every l other than $k-1$ and k, and of an entire component of Λ_{k-1} (with subscripts taken modulo K). Call agent $i \in N_k$ an *inner hero* if he gets the votes of all members of $N_k \setminus \{i\}$.

Define the set X of *eligible agents* as follows. If there exists at least one outer hero, it is the set of outer heroes. If there is no outer hero, it is the set of inner heroes.

If $X = \emptyset$, then $\varphi(x) = i^*$. If $|X| = 1$, say $X = \{i\}$, then $\varphi(x) = i$.

Step 2. If $|X| \geq 2$, let $Y = \arg\max_{i \in X} s_i(N \setminus X)$. Using a fixed priority ordering of the set of agents N to break ties, the winner $\varphi(x)$ is the member of Y with the highest priority among members of Y.

[15] By this we mean that the sizes differ by at most 1.

We give an example[16] that illustrates the above definition. Say there are 14 agents, partitioned into three districts $N_1 = \{1, 2, 3, 4, 5\}$, $N_2 = \{6, 7, 8, 9\}$, and $N_3 = \{10, 11, 12, 13\}$, with agent 14 serving as the default i^*. The agents in each district are further partitioned according to two attributes, age and gender. These partitions are represented in the following table:

N_1	Male	Female	N_2	Male	Female	N_3	Male	Female
Young	1, 2	3	Young	6	7	Young	10	11
Old	4	5	Old	8	9	Old	12	13

Here, in each district k, Π_k is the partition into the young and the old members, and Λ_k is the partition into the male and the female members of that district. The orthogonality of the partitions is witnessed by the fact that all entries in the table are nonempty. Consider now an agent in N_1, say. To qualify as an outer hero, he needs the votes of an entire age-group in N_2, and an entire gender-group in N_3 (e.g., if agents 6, 7, 11, and 13 all vote for 1, then 1 is an outer hero). To qualify as an inner hero, he needs the votes of all other members of his district (so, if agents 2, 3, 4, and 5 all vote for 1, then 1 is an inner hero).

THEOREM 2: *Cross-partition methods satisfy Monotonicity, Impartiality, and Full Pivots.*

PROOF: We note first some useful facts about heroes. Suppose there is an outer hero in district k. Then there can be no heroes outside N_k: outer heroes are ruled out by the orthogonality of the Π_l, Λ_l, and inner heroes are impossible since an outer hero needs the votes of a full component (hence at least two votes) from every other district. On the other hand, in district k itself, there may be one other outer hero and/or an inner hero. But, by definition, such an inner hero is not considered eligible.

To illustrate the above, consider the 14-agent example just given, and suppose that agent 1 is an outer hero by getting the votes of 6, 7, 11, and 13. Then there can be no outer hero in N_2, because the votes of 11 and 13 are already accounted for, and without them one cannot form an entire age-group in N_3. Similarly, there cannot be an outer hero in N_3, because the votes of 6 and 7 are not available, and without them one cannot form an entire gender-group in N_2. Furthermore, there can be no inner hero in either N_2 or N_3, because in each of these districts, two votes are already committed to agent 1. On the other hand, another member of N_1, say agent 5, may be a hero as well as 1 (she can be an outer hero due to the votes of 8, 9, 10, and 12, or she can be an inner hero due to the votes of 1, 2, 3, and 4).

[16]We are indebted to an anonymous referee for suggesting this example.

Impartiality for i^* is obvious: if she wins, this is determined in step 1, in which her vote plays no role. Next, note that, for agent i in N_k, being eligible does not depend upon her own vote. This is clear if she is an outer hero, and if she is an inner hero it is because, regardless of her vote, there can be no outer hero outside N_k. Moreover, an eligible agent cannot change the set of eligible agents: this is clear for an inner hero, and for an outer hero it follows because there is no outer hero outside her district. Together, these two facts establish Impartiality.

To show Full Pivots, consider a triple i, j, j'. Assume first that one of j, j' is i^*, say $j = i^*$. Then i is pivotal for i^*, j' at a profile where there are no heroes other than j', and the status of j' as a hero (inner if i, j' are in the same district, outer otherwise) depends on the vote of i. Next, assume that j, j' are in the same district N_k. Then we can have both of them outer heroes, and make sure that they get nearly the same total number of votes from agents in the other districts (this relies on the assumption that the components of Π_l, Λ_l are of nearly equal size). Now, if i is in a district other than N_k, he can be pivotal by disqualifying j or j', and otherwise he can be pivotal in step 2. Finally, if j, j' are in different districts, we can make them the only two inner heroes. Now, if i is in one of their districts, he can be pivotal by disqualifying j or j', and otherwise he can be pivotal in step 2.[17]

For Monotonicity, suppose initially i wins, then j changes his vote to i. If $i = i^*$, the eligible set remains empty after the change and we are done. Assume $i \neq i^*$ from now on. Whether i was initially an inner or an outer hero (or both), this remains true after the change and no other agent becomes a hero. If i was an outer hero, he was either the only eligible agent or one of two. Whether or not the change disqualifies the other one (if any), he remains the winner. If i was an inner hero, there was no outer hero to start with; the change could make i an outer hero as well, in which case i wins at once because he is the sole outer hero. Alternatively, the change could disqualify an inner hero in j's district, say j'. In this case, consider the effect of the change on step 2: agent j' turns from candidate to voter, and his vote may go to an opponent of i, but i now gets the vote of j, which guarantees that he still wins. Finally, if the eligible set does not change, agent i remains the winner. *Q.E.D.*

Note that, as in partition methods, one can replace the local unanimity requirement for an inner hero by an absolute quota q that is slightly higher than half the size of the district (enough to prevent the coexistence of an outer hero and an inner hero in distinct districts).

The cross-partition methods are too complex to qualify as practical award rules. By contrast, the partition methods are natural, and similar in spirit to

[17] To justify the last statement, we have to rule out a situation where j, say, is assured to win in step 2 by virtue of his local votes. The near equality of district sizes certainly suffices to achieve this. But note that we made this assumption to keep the presentation simple; a much weaker condition forbidding a district to hold a majority of all agents would suffice.

two-tier voting systems that are used in practice (except for the role of the default agent), and their normative performance is almost as convincing.

REMARK: Our partition methods distribute coalitional power in much the same way as the ordinary plurality rule. Any coalition S holding a strict majority in each district controls the outcome entirely (i wins if everyone in S nominates her); loosely speaking, any strict majority coalition is a strict majority in at least one district, and can make anyone in that district the winner in the same way. Similarly, in a cross-partition method, a coalition containing a majority of districts can prevent outer heroes, and make anyone in its districts the winner.[18]

3. TWO IMPOSSIBILITY RESULTS

We identify two systematic shortcomings of impartial nomination rules, already apparent in the various examples discussed in Sections 1.2 and 2.

First, consider the following property of nomination rules, requiring that the determination of the winner at a profile of nominations x should depend only on the profile of scores $s = \delta(x)$:

- *Anonymous Ballots*: for all $x, y \in N^N_-$,

$$\delta(x) = \delta(y) \quad \Rightarrow \quad \varphi(x) = \varphi(y).$$

This property does not require that agents be treated equally as candidates, only as voters. It means that nominations need not be signed and can be collected anonymously into one ballot box. The standard plurality rule with ties broken by a fixed priority ordering satisfies Anonymous Ballots. But, for instance, the partition methods violate Anonymous Ballots in that the agents are split into districts, and the method treats differently nominations from one's own district and those from other districts.

It turns out that there is a fundamental incompatibility between Anonymous Ballots and Impartiality. The intuition is that Impartiality does not allow i's ballot to affect i's winning; but if i's ballot is to be treated like anyone's ballot, this implies that it cannot affect anyone's winning.

THEOREM 3: *The only impartial nomination rules satisfying Anonymous Ballots are the constant rules*: *for some* i, $\varphi(x) = i$ *for all* $x \in N^N_-$.

PROOF: The set $\delta(N^N_-)$ of feasible profiles of scores is

$$S = \left\{ (s_1, \dots, s_n) \in \{0, 1, \dots, n-1\}^n \,\middle|\, \sum_{i=1}^{n} s_i = n \right\}.$$

[18]Thus if agents had standard linear preferences over all winners, cooperative instability (empty core) would arise from the usual patterns of cyclical preferences.

Anonymity says that our rule is defined directly on S, so we write it as $\varphi(s)$. We show next that if $s, s' \in S$ and $s_k = s'_k$ for some agent k, then $\varphi(s) = k \Leftrightarrow \varphi(s') = k$. Since s' is obtained from s by a redistribution of the scores of agents in $N \setminus \{k\}$, it suffices to prove this in the case when it is obtained by the transfer of one vote from i to i', for some $i, i' \in N \setminus \{k\}$. Assume first that $\max_{j\in N\setminus\{i,i',k\}} s_j \leq n-2$. Then there exists $x \in N_-^N$ with $\delta(x) = s$ and $x_k = i$ (this can be shown, e.g., by using a harem version of the marriage lemma). When k switches his vote to $x'_k = i'$, we have $\delta(x') = s'$, and Impartiality gives $\varphi(s) = k \Leftrightarrow \varphi(s') = k$. It remains to handle the case where, for some $j \in N \setminus \{i, i', k\}$, we have $s_j = s'_j = n-1$ and $s_i = s'_{i'} = 1$. In this case, consider $s'' \in S$ with $s''_j = n-2$, $s''_i = s''_{i'} = 1$. Note that s is obtained from s'' by transferring one vote from i' to j, while s' is obtained from s'' by transferring one vote from i to j. Since all entries in s'' are at most $n-2$, we can apply the conclusion of the previous case twice to obtain $\varphi(s'') = k \Leftrightarrow \varphi(s) = k$ and $\varphi(s'') = k \Leftrightarrow \varphi(s') = k$, and deduce that $\varphi(s) = k \Leftrightarrow \varphi(s') = k$. So for every agent, winning only depends on his own score.

Now, consider $s^* = (1, 1, \dots, 1)$ and say $\varphi(s^*) = i$. We claim that $\varphi(s) = i$ for every $s \in S$, making φ a constant rule. Indeed, suppose for the sake of contradiction that $s \in S$ and $\varphi(s) = j \neq i$. We can find $s' \in S$ with $s'_i = 1$ and $s'_j = s_j$ (the other entries of s' are chosen arbitrarily to make the sum of all entries n). We apply the conclusion of the previous paragraph twice: since $s'_i = s^*_i$ and $\varphi(s^*) = i$, we must have $\varphi(s') = i$; since $s'_j = s_j$ and $\varphi(s) = j$, we must have $\varphi(s') = j$. This contradiction completes the proof. *Q.E.D.*

Our next result is a more serious limitation of impartial nomination rules. All such rules must disregard at some profiles the unanimous views of the agents in favor or against a certain agent.

THEOREM 4: *There exists no nomination rule that satisfies Impartiality*, *Positive Unanimity*, *and Negative Unanimity*.

Since Positive Unanimity is a consequence of No Exclusion, Impartiality, and Monotonicity, Theorem 4 immediately implies the following.

COROLLARY 1: *There exists no nomination rule that satisfies Impartiality*, *No Exclusion*, *Negative Unanimity*, *and Monotonicity*.

The theorem and its corollary explain why winning by default (in some circumstances) is an inevitable ingredient of all our constructions of impartial nomination rules.

The difficult proof of Theorem 4 is the subject of Section 4 below. To conclude this section, we show that each property in the theorem and its corollary is required for the impossibility statement. The ordinary plurality rule meets all properties except Impartiality. The rule Maj^{i^*} meets all properties except Negative Unanimity. The rule Maj_{i^0} meets all properties except Positive Unanimity in the theorem, and No Exclusion in the corollary.

Finally, Monotonicity is needed[19] in the corollary; in other words, Theorem 4 does not remain true if Positive Unanimity is weakened to No Exclusion. This fact requires a new and nontrivial construction.

EXAMPLE 1—A Nomination Rule for $n \geq 5$ That Satisfies Impartiality, No Exclusion, No Dummy, and Negative Unanimity: To construct this rule, we view the set of agents as a linearly ordered set, say $N = \{1, \dots, n\}$ with the natural ordering. Given a profile of nominations $x = (x_i)_{i \in N}$, the winner $\varphi(x)$ is determined according to the following cases:

(a) If there exist agents $i < j < k < n$ such that $x_i = x_j = k$, and $x_\ell = n$ for all $\ell \in N \setminus \{i, j, k, n\}$, then $\varphi(x) = n$.

(b) Else, if there exists at least one agent $q < n$ such that the set

$$\left\{p \in N \setminus \{x_n\} | p < q, x_p = q\right\}$$

is nonempty, then $\varphi(x)$ is the first such q.

(c) Else, $\varphi(x) = x_n$.

The verification of No Exclusion, No Dummy, and Negative Unanimity is easy (bearing in mind that $n \geq 5$). To check Impartiality, assume for the sake of contradiction that $\varphi(x) \neq \varphi(x')$ for some two profiles x, x' which differ only in the vote of one agent, and that agent is $\varphi(x)$ or $\varphi(x')$. This is clearly impossible if x and x' fall under the same case (among (a), (b), and (c)), because the determination of the winner in each case is independent of his vote. Next, assume that exactly one of the profiles, say x, falls under (a); let $i < j < k$ witness this. Then $\varphi(x) = n$, and n cannot move the profile out of (a), so we only need to check if $\varphi(x')$ can be the mover. If x' falls under (b), so $\varphi(x') = q$ as in (b), then we must have $q \geq k$, because agents before k do not get any votes from preceding agents; but $q = k$ is impossible because k cannot move out of (a), while $q > k$ results in k being an earlier agent than q for whom the set in (b) is nonempty (both i and j precede k and vote for him, and at least one of them is not x'_n). If x' falls under (c), so $\varphi(x') = x'_n$, then a change in the vote of x'_n must leave k with at least one vote from a preceding agent (either i or j) who is not x'_n, thus placing x' under (b) rather than (c). Finally, assume that one of the profiles, say x, falls under (b), and the other, x', falls under (c). Then $\varphi(x)$ is determined as in (b), and this determination is independent of the vote of $\varphi(x)$, as remarked earlier, and also of the vote of $\varphi(x')$, because the latter is $x'_n = x_n$ and his vote is ignored in (b).

[19] This is the case for every $n \geq 5$. For $n = 4$, it can be shown that Impartiality, No Exclusion, and Negative Unanimity already lead to impossibility.

Thus the rule φ has all the properties stated in Example 1. Of course, Monotonicity and Positive Unanimity fail for agent n, because he only wins under (a), which requires that certain agents (i and j) *not* vote for him.

4. PROOF OF THEOREM 4

We consider an extension of the class of nomination rules, allowing the prize to be awarded by a lottery. We denote by $\Delta(N)$ the set of lotteries over N, that is,

$$\Delta(N) = \left\{\delta \in \mathbb{R}^N \,\middle|\, \delta_i \geq 0 \ \forall i \in N, \sum_{i \in N} \delta_i = 1\right\}.$$

DEFINITION 4: A randomized nomination rule is a function $\varphi : N_-^N \to \Delta(N)$.

The interpretation is clear: given a profile of nominations x, the prize is awarded to agent i with probability $\varphi_i(x)$. The class of randomized nomination rules contains that of nomination rules, as we identify the deterministic award of the prize to agent i with the degenerate lottery concentrated on i. Some of the properties defined earlier for nomination rules extend naturally to randomized ones, specifically:

- *Impartiality*: for all $i \in N$, $x_i, x_i' \in N \setminus \{i\}$, and all $x_{-i} \in N_-^{N \setminus i}$,

$$\varphi_i(x_i, x_{-i}) = \varphi_i(x_i', x_{-i}).$$

- *Negative Unanimity*: for all $x \in N_-^N$ and all $i \in N$,

$$s_i = 0 \quad \Rightarrow \quad \varphi_i(x) = 0.$$

For the next property, we consider the set S_N of permutations of N. This set acts on the set N_-^N of nomination profiles in a natural way: $\sigma \in S_N$ transforms profile x to a new profile x^σ, so that whenever i nominates j in x, $\sigma(i)$ nominates $\sigma(j)$ in x^σ (in compact notation, $(x^\sigma)_k = \sigma(x_{\sigma^{-1}(k)})$). In other words, x^σ arises from x by renaming the agents according to σ. The following axiom requires that the rule be invariant with respect to such renaming:

- *Symmetry*: for all $\sigma \in S_N$, $x \in N_-^N$, and all $i \in N$,

$$\varphi_{\sigma(i)}(x^\sigma) = \varphi_i(x).$$

A deterministic nomination rule cannot be symmetric.[20] The advantage of working with randomized nomination rules is that, essentially, we may assume

[20] To see this, take a cyclic $\sigma \in S_N$, and consider the profile x, where $x_i = \sigma(i)$ for all $i \in N$. Then $x^\sigma = x$, hence Symmetry requires $\varphi_{\sigma(i)}(x) = \varphi_i(x)$ for all $i \in N$, and therefore $\varphi(x) = (1/n, \dots, 1/n)$.

Symmetry without loss of generality. Indeed, the operation of symmetrization, which consists of averaging over all possible renamings of the agents, turns any randomized nomination rule φ (and, in particular, any deterministic one) into a symmetric one φ^{sym}, namely:

$$\varphi_i^{\mathrm{sym}}(x) = \frac{1}{n!} \sum_{\sigma \in S_N} \varphi_{\sigma(i)}\big(x^\sigma\big) \quad \text{for all } i \in N.$$

We state now our main result for randomized nomination rules, and deduce Theorem 4 from it.

PROPOSITION 1: *Let φ be a randomized nomination rule that satisfies Impartiality, Negative Unanimity, and Symmetry. Let $i \in N$, and let $x \in N_-^N$ be such that $s_i = n - 1$. Then $\varphi_i(x) = \frac{n-1}{n}$.*

PROOF OF THEOREM 4: Assume, for the sake of contradiction, that φ is a nomination rule satisfying Impartiality, Positive Unanimity and Negative Unanimity. Symmetrizing φ, we obtain a randomized nomination rule φ^{sym}. It is easy to check that, in addition to being symmetric, φ^{sym} inherits from φ the properties of Impartiality and Negative Unanimity. Moreover, Positive Unanimity of φ implies that if $s_i = n-1$, then $\varphi_i^{\mathrm{sym}}(x)$ must be 1, whereas by Proposition 1 it has to be $\frac{n-1}{n}$, a contradiction. *Q.E.D.*

The symmetrization argument above also implies that the statement of Theorem 4 holds for randomized rules as well.

It remains to prove Proposition 1. As the proof is quite involved, we begin by describing it for the case of four agents. The case $n = 4$ is much more transparent than the general case, yet it illustrates some of the ideas that appear in the general proof. We represent nomination profiles by nomination graphs: directed graphs with n nodes and exactly one arc incident from each node (and pointing to that node's nominee). A symmetric randomized nomination rule φ assigns to the nodes of any such graph their winning probabilities, and this assignment does not depend on the labeling of the nodes by agents. Hence we move freely between graphs in which all, part, or none of the nodes are labeled by agents. Consider, for $n = 4$, the graphs in Figure 1.

Our goal is to show that if φ satisfies the three properties in Proposition 1, then $\varphi_i(d)$, the probability assigned to node i in Figure 1(d), must equal $\frac{3}{4}$. We start with Figure 1(a): by Symmetry, all probabilities must be equal, and in particular, $\varphi_j(a) = \frac{1}{4}$. Since agent j can change his vote to obtain Figure 1(c), Impartiality implies that $\varphi_j(c) = \varphi_j(a) = \frac{1}{4}$. Next, we consider Figure 1(b): by Negative Unanimity, only the two nodes forming a cycle may win, and by Symmetry, their winning probabilities are equal, hence $\varphi_i(b) = \frac{1}{2}$. Since agent i can move to Figure 1(c), Impartiality implies that $\varphi_i(c) = \varphi_i(b) = \frac{1}{2}$. Considering now Figure 1(c), by Negative Unanimity only i, j, k may win,

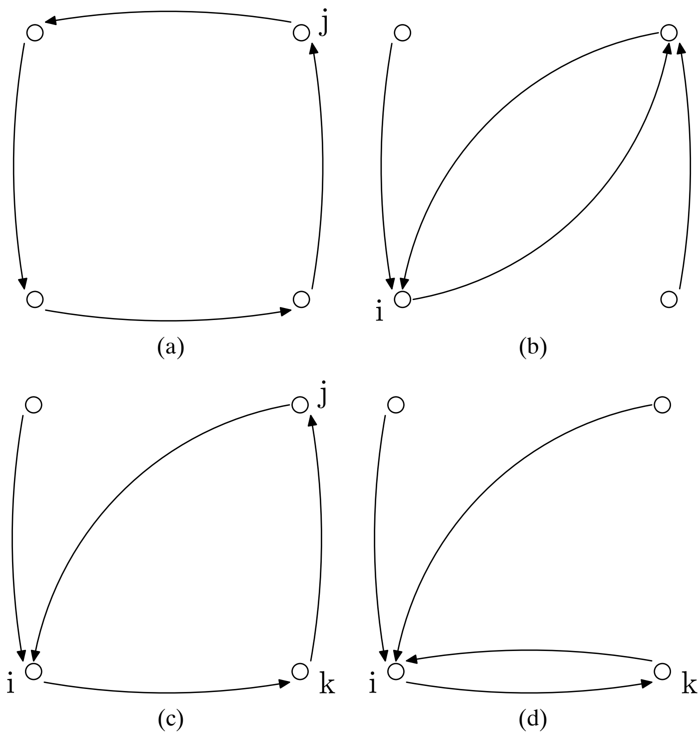

FIGURE 1.—Some nomination graphs for four agents.

so having deduced the probabilities $\varphi_j(c) = \frac{1}{4}$, $\varphi_i(c) = \frac{1}{2}$, we conclude that $\varphi_k(c) = 1 - (\frac{1}{4} + \frac{1}{2}) = \frac{1}{4}$. But agent k can move from Figure 1(c) to 1(d), hence by Impartiality, $\varphi_k(d) = \varphi_k(c) = \frac{1}{4}$. Finally, in Figure 1(d), by Negative Unanimity, only i, k may win, so $\varphi_i(d) = 1 - \frac{1}{4} = \frac{3}{4}$, as required.

Before starting the actual proof of Proposition 1, we introduce the nomination profiles (graphs) that appear in it, and establish some notation and simple facts about them. We work only with a subclass of all nomination graphs; a graph is in this subclass if it contains a directed cycle of some length r (necessarily $2 \le r \le n$), and from each of the nodes outside this cycle there is an arc to a node in the cycle. We refer to the nodes in and out of the cycle as insiders and outsiders, respectively.

For a graph in this subclass and a given insider i, we can describe the graph from the point of view of i by an r-tuple $(a_1, \dots, a_r)$, where a_d is the number of outsiders sending arcs to the insider in dth position after i along the cycle. For example, Figure 1(a) is described from the point of view of each insider by $(0, 0, 0, 0)$; Figure 1(b) by $(1, 1)$; Figure 1(c) from i's point of view by $(0, 0, 1)$, from k's by $(0, 1, 0)$, and from j's by $(1, 0, 0)$; Figure 1(d) from i's point of view by $(0, 2)$, and from k's by $(2, 0)$. Note that, in every such representation, we have $\sum_{d=1}^{r} a_d = n - r$, and of course, each a_d is a nonnegative integer. We denote by $A_r(n)$ the set of all r-tuples $(a_1, \dots, a_r)$ satisfying these constraints.

Using the randomized nomination rule φ, for $(a_1, \dots, a_r) \in A_r(n)$ we denote by $\psi(a_1, \dots, a_r)$ the winning probability of an insider, given that $(a_1, \dots, a_r)$ describes the graph from his point of view. By Symmetry of φ, this is well defined. To illustrate, the proof for $n = 4$ established, using Figures 1(a)–(d), that

a. $\psi(0, 0, 0, 0) = \frac{1}{4}$,
b. $\psi(1, 1) = \frac{1}{2}$,
c. $\psi(1, 0, 0) = \frac{1}{4}$, $\psi(0, 0, 1) = \frac{1}{2}$, $\psi(0, 1, 0) = \frac{1}{4}$,
d. $\psi(2, 0) = \frac{1}{4}$, $\psi(0, 2) = \frac{3}{4}$.

If $(a_1, \dots, a_r)$ describes the graph from a certain insider's point of view, then $(a_2, \dots, a_r, a_1)$ describes the same graph from the next insider's point of view, $(a_3, \dots, a_r, a_1, a_2)$ from the following one's, and so on. Since, by Negative Unanimity, only insiders may win, we obtain the identity

$$(1) \qquad \psi(a_1, \dots, a_r) + \psi(a_2, \dots, a_r, a_1) + \cdots + \psi(a_r, a_1, \dots, a_{r-1}) = 1.$$

PROOF OF PROPOSITION 1: Let φ be as in the proposition, and let ψ be the function derived from it as explained above. We have to show that $\psi(0, n-2) = \frac{n-1}{n}$.

We begin by computing the sum $\sum_{(a_1,\dots,a_r)\in A_r(n)} \psi(a_1, \dots, a_r)$ for any fixed $2 \leq r \leq n$. Consider the following equivalence relation over $A_r(n)$: $(a_1, \dots, a_r) \sim (b_1, \dots, b_r)$ if $(b_1, \dots, b_r)$ is obtained from $(a_1, \dots, a_r)$ by a cyclic shift of coordinates (e.g., $(0, 1, 1, 3, 0, 1, 1, 3) \sim (1, 3, 0, 1, 1, 3, 0, 1)$ in $A_8(18)$). Let $\mathcal{E}_r(n)$ be the set of equivalence classes of $A_r(n)$ under $\sim$. Note that if $(a_1, \dots, a_r) \in C \in \mathcal{E}_r(n)$, then $|C|$ divides r, and each r-tuple in C appears in (1) exactly $r/|C|$ times. Thus (1) may be rewritten as $\frac{r}{|C|} \sum_{(a_1,\dots,a_r)\in C} \psi(a_1, \dots, a_r) = 1$. Using this, we compute

$$(2) \qquad \sum_{(a_1,\dots,a_r)\in A_r(n)} \psi(a_1, \dots, a_r) = \sum_{C\in\mathcal{E}_r(n)} \sum_{(a_1,\dots,a_r)\in C} \psi(a_1, \dots, a_r) = \sum_{C\in\mathcal{E}_r(n)} \frac{|C|}{r}$$
$$= \frac{1}{r}\left|A_r(n)\right| = \frac{1}{r}\binom{n-1}{r-1} = \frac{1}{n}\binom{n}{r},$$

where the last two equalities follow from a well-known counting argument for the size of $A_r(n)$, and a simple calculation of binomial coefficients.

Next, we observe that every r-tuple in $A_r(n)$ with a positive first coordinate corresponds in a one-to-one way to an $(r + 1)$-tuple in $A_{r+1}(n)$ with first coordinate zero; the pairing is given by

$$(a_1, a_2, \dots, a_r) \quad \leftrightarrow \quad (0, a_1 - 1, a_2, \dots, a_r).$$

The crucial fact about this pairing is that an insider can move from the graph described by one member of the pair to that described by the other member, both from his point of view, by changing his vote (indeed, to move from left to right, an insider has to replace his vote for the next insider by a vote for an outsider who supports the latter). Impartiality implies that we have, for the pairing above,

$$\psi(a_1, a_2, \dots, a_r) = \psi(0, a_1 - 1, a_2, \dots, a_r). \tag{3}$$

The collection of all these pairs, for all $2 \le r \le n-1$, exactly covers $(\bigcup_{r=2}^{n} A_r(n)) \setminus \{(0, n-2)\}$. Moreover, in each pair, one member has an even number of coordinates and the other an odd number. Summing up all the sides of (3) with an even number of coordinates on the one hand, and all those with an odd number on the other, gives the equality

$$\begin{aligned}
&\sum_{(a_1,\dots,a_r)\in(\bigcup_{r\,\text{even}} A_r(n))\setminus\{(0,n-2)\}} \psi(a_1, \dots, a_r) \\
&\quad = \sum_{(a_1,\dots,a_r)\in\bigcup_{r\,\text{odd}} A_r(n)} \psi(a_1, \dots, a_r).
\end{aligned}$$

From here, we can compute the value of $\psi(0, n-2)$ with the help of (2):

$$\begin{aligned}
\psi(0, n-2) &= \sum_{(a_1,\dots,a_r)\in\bigcup_{r\,\text{even}} A_r(n)} \psi(a_1, \dots, a_r) \\
&\quad - \sum_{(a_1,\dots,a_r)\in\bigcup_{r\,\text{odd}} A_r(n)} \psi(a_1, \dots, a_r) \\
&= \sum_{r=2}^{n} (-1)^r \sum_{(a_1,\dots,a_r)\in A_r(n)} \psi(a_1, \dots, a_r) = \sum_{r=2}^{n} (-1)^r \frac{1}{n}\binom{n}{r} \\
&= \frac{1}{n}\left(\sum_{r=0}^{n} (-1)^r \binom{n}{r} - \binom{n}{0} + \binom{n}{1}\right) = \frac{n-1}{n}. \qquad \textit{Q.E.D.}
\end{aligned}$$

The proof of our main impossibility result—Theorem 4, and its corollary—is now complete.

Our final remark concerns Proposition 1. In the class of randomized nomination rules, the rule that assigns winning probabilities that are proportional to the number of nominations is the most natural one. We call this rule *random dictatorship*: an agent is chosen uniformly at random, and his nominee gets the prize. If a rule satisfies Impartiality, Negative Unanimity, and Symmetry, Proposition 1 says that it must coincide with random dictatorship at those profiles where an agent is nominated by all others. One might ask if this is true at all profiles, namely, if random dictatorship is characterized by these three axioms. This is the case for $n \le 4$ (we omit the uninformative proof), but not beyond that, as indicated by the following example.

EXAMPLE 2—A Randomized Nomination Rule for Five Agents That Satisfies Impartiality, Negative Unanimity, and Symmetry, but Is not Random Dictatorship: We take the nomination rule φ defined in Example 1, and symmetrize it to obtain φ^{sym}. Since φ satisfies Impartiality and Negative Unanimity, φ^{sym} satisfies these properties and of course Symmetry. To see that φ^{sym} is not random dictatorship, consider a nomination graph that consists of two disjoint directed cycles of lengths 2 and 3, respectively. For any labeling of the nodes by agents numbered $1, \dots, 5$, the profile does not fall under case (a) of the definition of φ, because no agent gets two nominations. Moreover, the profile always falls under case (b), because each cycle must contain an arc with increasing labels, and one of the cycles does not contain the label 5; an increasing arc in such a cycle provides the condition of (b). In fact, an agent q as in (b) exists only in the cycle that does not contain 5, so the winner always belongs to that cycle. It follows that in φ^{sym}, the winning probability is $\frac{3}{10}$ for each node on the 2-cycle, and $\frac{2}{15}$ for each node on the 3-cycle; by contrast, in random dictatorship, the winning probability is $\frac{1}{5}$ for everyone.

5. AWARD RULES WITH ABSTRACT MESSAGES

We briefly discuss what can be achieved if, instead of nominations, the agents have arbitrary message spaces. In this model, a message is abstract, and has no natural interpretation as to who should win the prize.

The set of agents is N, as before. Each agent $i \in N$ is endowed with a message space M^i, and we write $M^N = \bigtimes_{i \in N} M^i$ for the set of message profiles.

DEFINITION 5: An award rule is a function $f : M^N \to N$.

The formulations of Impartiality, No Dummy, and No Exclusion extend in a straightforward way to award rules—replacing nominations and nomination profiles by messages and message profiles; we do not restate them here.[21]

For four or more agents, there exist award rules satisfying the above three properties, whereas for two or three agents, Impartiality is incompatible with *each one* of No Exclusion and No Dummy. The following proposition is essentially due to Kato and Ohseto (2004), who stated it in the context of pure exchange economies (see item 3 in Section 1.4).

PROPOSITION 2: (i) *If* $n \leq 3$, *an impartial award rule is either constant* (*the same agent wins no matter what*), *or has one agent choosing the winner among the other two*.

(ii) *If* $n \geq 4$, *there are impartial award rules meeting No Exclusion and No Dummy*.

[21] But none of Monotonicity, Negative Unanimity and Positive Unanimity, can be defined in the abstract setting of award rules.

PROOF: (i) The case $n = 2$ is very easy, and left to the reader. Now set $N = \{1, 2, 3\}$. If the rule f is not constant, then some agent, say 1, can change the outcome at some message profile, say (m_2, m_3), of the other agents, by changing his message. Keeping this (m_2, m_3) fixed and varying m_1, the winner $f(m_1, m_2, m_3)$ is never 1 (by Impartiality), so we get a partition of M^1 in two nonempty parts $M^1(2)$ and $M^1(3)$, such that

$$f(m_1, m_2, m_3) = j \quad \text{if} \quad m_1 \in M^1(j), \quad j = 2, 3.$$

Note that, for all $m_1 \in M^1(2)$ and $m'_3 \in M^3$, we cannot have $f(m_1, m_2, m'_3) = 1$; otherwise, by Impartiality, $f(m'_1, m_2, m'_3) = 1$ as well for any $m'_1 \in M^1(3)$, a contradiction because $f(m'_1, m_2, m_3) = 3$. Impartiality rules out $f(m_1, m_2, m'_3) = 3$ as well, so we have $f(M^1(2) \times \{m_2\} \times M^3) = 2$; one more application of the property gives $f(M^1(2) \times M^2 \times M^3) = 2$, and a symmetrical argument gives $f(M^1(3) \times M^2 \times M^3) = 3$.

(ii) We describe an award rule for $n \geq 4$, adapted from Kato and Ohseto (2004). Set $N = N_0 \cup \{i^*\}$, and arrange $N_0 = \{1, 2, \ldots, n-1\}$ in this order along a circle (so their numbers are taken modulo $n-1$). Let $M^i = \{0, 1\}$ for all $i \in N$, and, for a given $m \in M^N$, denote $\operatorname{supp}_0(m) = \{i \in N_0 | m_i = 1\}$. The winner is determined as follows. If $\operatorname{supp}_0(m) = \{i\}$, then $i - 1$ wins if $m_{i^*} = 0$, and $i + 1$ wins if $m_{i^*} = 1$. If $\operatorname{supp}_0(m) = \{i, i + 1\}$, then i wins if $m_{i^*} = 0$, and $i + 1$ wins if $m_{i^*} = 1$. In all other cases, i^* wins. It is easy to check that this rule is impartial and satisfies No Exclusion and No Dummy. *Q.E.D.*

It is interesting to note that the possibility part of Proposition 2 was established using the smallest conceivable message spaces: $|M^i| = 2$ for all i. Therefore, the impossibility/possibility frontier uncovered by Proposition 2 is independent of the specification of message spaces.

6. CONCLUDING COMMENTS

1. Beyond a single nomination for the prize, we can think of allowing more informative messages in at least two ways. If an agent is allowed to nominate (approve of) one or more other agents, without ranking them, we do not see how to process this information in a way similar to our partition methods, a question worthy of further study. On the other hand, any method that aggregates nominations of one or more agents must, in particular, aggregate nominations of single agents, and as such is subject to our negative results.

A second natural message is a complete preference over one's peers, describing the top choice for winner, the second best choice, and so on. We can then speak of a *voting rule*, completely similar to a standard voting method. Our partition methods can be adapted to aggregate these more refined opinions. Given, for each agent i, a ranking of $N \setminus \{i\}$, we tally the votes in two steps. In step 1, the top choice of i in her district is her local vote, and local winners are chosen from these votes; in step 2, we collect from each agent who is not a local winner her top choice among the local winners, and use those votes to choose a global winner among local winners. Another open question is to determine the possibility frontier of this new class of award rules, in terms of normative properties relevant to efficiency and fairness.

2. The randomized nomination rules used in the proof of Theorem 4 are effectively dividing one unit of homogeneous commodity (the probability of winning) between our agents. Thus they are related to the dollar division methods in de Clippel, Moulin, and Tideman (2008), but with much coarser individual messages. We can adapt some of the methods discussed there to the randomized voting context, when individual messages are a nomination, or a full ordering of the peers. These rules are worthy of further study.

3. There is an escape route from the impossibility of Theorem 3, which consists in restricting the domains of permissible nominations. According to this approach, each agent i is restricted to nominate only agents in some (predetermined) nonempty subset $M^i \subseteq N \setminus \{i\}$. We were able to show that, with a judicious choice of these subsets M^i based on a tree structure on the set of agents, one may define impartial restricted nomination rules, called median methods, that do satisfy the analog of Anonymous Ballots as well as other desirable properties. Unfortunately, the restrictions imposed by these methods and their asymmetric nature limit their applicability. Hence we do not pursue this approach here; the interested reader can find the details in Holzman and Moulin (2010).

4. It would be interesting to try to extend our work to the problem of awarding k identical prizes, where k is fixed and may be greater than 1. This problem was studied in Alon, Fischer, Procaccia, and Tennenholtz (2011), but with a different approach and different message spaces than ours (as explained in Section 1.4). Perhaps the most natural analog of our nomination rules would have each agent nominate k other agents for the prize. It is quite straightforward to extend our main axioms to such a model. But the problem becomes more complex, and it is not at all clear how to generalize our results.

Alternatively, we may consider profiles of single nominations as in this paper, and rules that select k winners. Our partition methods may be adapted to this setting by using an ordered list of k default agents, so that if there are only $\ell < k$ local winners, the first $k - \ell$ default agents join the winners. As to our negative results, it is not difficult to generalize Theorem 3 to this setting, but we do not know about Theorem 4.

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