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Independence—domination duality

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Abstract

Given a system $\mathcal{G} = (G_1, G_2, \dots, G_m)$ of m graphs on the same vertex set V, define the "joint independence number" $\alpha_\cap(\mathcal{G})$ as the maximal size of a set which is independent in all graphs G_i. Let also $\gamma_\cup(\mathcal{G})$ be the "collective domination number" of the system, which is the minimal number of neighborhoods, each taken from any of the graphs G_i, whose union is V. König's classical duality theorem can be stated as saying that if $m = 2$ and both graphs G_1, G_2 are unions of disjoint cliques then $\alpha_\cap(G_1, G_2) = \gamma_\cup(G_1, G_2)$. We prove that a fractional relaxation of $\alpha_\cap$, denoted by $\alpha^*_\cap$, satisfies the condition $\alpha^*_\cap(G_1, G_2) \geqslant \gamma_\cup(G_1, G_2)$ for any two graphs G_1, G_2, and $\alpha^*_\cap(G_1, G_2, \dots, G_m) > \frac{2}{m}\gamma_\cup(G_1, G_2, \dots, G_m)$ for any $m > 2$ and all graphs $G_1, G_2, \dots, G_m$. We prove that the convex hull of the (characteristic vectors of the) independent sets of a graph contains the anti-blocker of the convex hull of the non-punctured neighborhoods of the graph and vice versa. This, in turn, yields $\alpha^*_\cap(G_1, G_2, \dots, G_m) \geqslant \gamma^*_\cup(G_1, G_2, \dots, G_m)$ as well as a dual result. All these results have extensions to general simplicial complexes, the graphical results being obtained from the special case of the complexes of independent sets of graphs.

Keywords: Graph systems; Independence; Domination; König's duality; Matroid intersection; Anti-blocker; Fractional ISR

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1. Introduction

Menger's [10] theorem from 1927 was the first combinatorial fact cast in min–max form. This theorem prompted König to formulate his own theorem, which was previously written [8] as a necessary and sufficient condition for the existence of a matching, as a min–max theorem:

Theorem 1.1. *In a bipartite graph the size of the largest matching is the minimal size of a vertex cover.*

Edmonds [5] realized that this theorem can be viewed as relating to two structures which are imposed on the same ground set. The ground set is the edge set of the bipartite graph, and the two structures are the adjacency relationship in one side of the graph and the adjacency relationship in the other side, respectively. In other words, the line graph of a bipartite graph is the union of two systems of disjoint cliques. Let us call a graph consisting of vertex disjoint cliques a *partition* graph. In this terminology, König's theorem can be stated as follows:

Theorem 1.2. *Given two partition graphs on the same vertex set V, the maximal size of a set which is independent in both graphs is equal to the minimal number of cliques, taken from any of the two graphs, whose union covers V.*

A natural question is what happens if we drop the demand on the two graphs, that is, what is true for two general graphs on the same vertex set. As stated above, the theorem is generally false even if the two graphs are the same, as seen, for example, by considering any minimally imperfect graph. But with another formulation, the theorem is more generalizable. In this formulation covering by cliques is replaced by domination. The motivation for studying this version comes from independent systems of representatives (ISRs), in which domination has played a key role (see Section 3).

To formulate our notions precisely, we shall need the following definitions.

A hypergraph $\mathcal{C}$ is called a *simplicial complex* if it is closed down, namely $\sigma \in \mathcal{C}$, $\tau \subseteq \sigma$ imply $\tau \in \mathcal{C}$. (Although we shall not be using topology, we prefer this topologically-oriented term to "closed down hypergraph" because it is shorter, and since related work on ISRs did use topological methods.) Henceforth we shall omit the adjective "simplicial." A set $\sigma \in \mathcal{C}$ will be called a *simplex*. Also, when we say that $\mathcal{C}$ is a complex on V, we mean that V is the union of all simplices in $\mathcal{C}$. For a graph G we denote by $\mathcal{I}(G)$ the complex of independent sets of G and by $\mathcal{N}(G)$ the complex of non-punctured neighborhoods of G, namely $\sigma \in \mathcal{N}(G)$ if there exists a vertex v such that all members of σ are either adjacent or equal to v. For a system $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$ of complexes on the same ground set V, let $\mu_\cap(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ denote the maximal size of a simplex belonging to $\bigcap_{i=1}^m \mathcal{C}_i$. Also write $\chi_\cup(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ for the minimal number of simplices from $\bigcup_{i=1}^m \mathcal{C}_i$ whose union is V. (The use of χ here is inspired by the standard notation for the chromatic number; it should not be confused with its other standard use, to denote the characteristic vector of a set, which we adopt elsewhere in the paper.) For a system $G_1, G_2, \dots, G_m$ of graphs on the same vertex set write $\alpha_\cap(G_1, G_2, \dots, G_m) = \mu_\cap(\mathcal{I}(G_1), \mathcal{I}(G_2), \dots, \mathcal{I}(G_m))$ and $\gamma_\cup(G_1, G_2, \dots, G_m) = \chi_\cup(\mathcal{N}(G_1), \mathcal{N}(G_2), \dots, \mathcal{N}(G_m))$. Equivalently,

$$\gamma_\cup(G_1, G_2, \dots, G_m) = \min\left\{ \sum_{i=1}^m |X_i| \;\middle|\; \bigcup_{i=1}^m N_{G_i}(X_i) = V \right\},$$

where $N_G(X)$ denotes the neighborhood of X in G, that is, the set of vertices that are either in X or have a neighbor in X. In this terminology König's theorem can be stated as follows:

Theorem 1.3. *For two partition graphs G_1 and G_2 on the same vertex set we have*

$$\alpha_\cap(G_1, G_2) = \gamma_\cup(G_1, G_2).$$

With this formulation, we still do not expect the assertion of equality to admit interesting generalizations (for example, if $G_1 = G_2$ is a star then $\alpha_\cap$ is large while $\gamma_\cup = 1$). But we are interested in the nontrivial part of König's theorem, which in our terminology is the inequality $\alpha_\cap(G_1, G_2) \geqslant \gamma_\cup(G_1, G_2)$.

This inequality is true if $G_1 = G_2$ is arbitrary, because a maximal independent set is dominating. More interestingly, using a topological method (see the argument in [2]) it is possible to show that the inequality is true if G_1 is a partition graph and G_2 is "stably wide," meaning that each induced subgraph H of G_2 contains an independent set, demanding as many vertices to dominate it as H itself. For examples of classes of graphs that are stably wide, see [1].

For general graphs, the inequality $\alpha_\cap(G_1, G_2) \geqslant \gamma_\cup(G_1, G_2)$ is false. For example, if G_1 is a path of length (number of edges) 3 and G_2 is its complement (which is also a path of length 3), then $\alpha_\cap(G_1, G_2) = 1$ while $\gamma_\cup(G_1, G_2) = 2$.

The main aim of this paper is to show that though the inequality fails in general, it is valid if we replace its α side by a fractional version. To define this relaxation, we introduce the following notation. For a complex $\mathcal{C}$ let $\Omega(\mathcal{C})$ be the polytope whose extreme points are the characteristic vectors of simplices from $\mathcal{C}$, that is, $\Omega(\mathcal{C}) = \mathrm{conv}\{\chi_\sigma \mid \sigma \in \mathcal{C}\}$. For a system $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of complexes write $\Omega(\mathcal{L}) = \bigcap_{i=1}^m \Omega(\mathcal{C}_i)$. Write $\mu_\cap^*(\mathcal{L}) = \max\{\vec{x} \cdot \mathbf{1} \mid \vec{x} \in \Omega(\mathcal{L})\}$, where $\mathbf{t}$ denotes the vector all of whose entries are equal to t. For a system $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ of m graphs write $\alpha_\cap^*(\mathcal{G}) = \mu_\cap^*(\mathcal{I}(G_1), \mathcal{I}(G_2), \ldots, \mathcal{I}(G_m))$. Our result for $m = 2$ is

Theorem 1.4. *For any two graphs G_1 and G_2 on the same vertex set we have*

$$\alpha_\cap^*(G_1, G_2) \geqslant \gamma_\cup(G_1, G_2).$$

As an illustration, for the two complementary paths of length 3 we have $\alpha_\cap^*(G_1, G_2) = 2$, since the vector $\mathbf{\frac{1}{2}}$ belongs to $\Omega(\mathcal{I}(G_1), \mathcal{I}(G_2))$. On the other hand, as we have already noted, $\gamma_\cup(G_1, G_2) = 2$.

Lovász [9] proved that in m-partite m-graphs one has $\tau \leqslant \frac{m}{2}\tau^*$, where τ denotes the minimal size of a vertex cover, and τ^* is its fractional relaxation. In our terminology this can be re-phrased as saying that for all $m \geqslant 2$ and any m partition graphs on the same vertex set one has $\alpha_\cap^* \geqslant \frac{2}{m}\gamma_\cup$. We shall give a common generalization of this and of Theorem 1.4:

Theorem 1.5. *For a system $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ of $m \geqslant 2$ graphs on the same vertex set we have*

$$\alpha_\cap^*(\mathcal{G}) \geqslant \frac{2}{m}\gamma_\cup(\mathcal{G}).$$

For $m > 2$ strict inequality holds.

We remark that the inequality in Theorem 1.5 is best possible, in the following sense. For $m = 2$ it may hold as an equality. For any fixed $m > 2$ there are examples where the ratio $\alpha_\cap^*/\gamma_\cup$ is arbitrarily close to $\frac{2}{m}$. Such examples were given in [3] in the setting of m-partite m-graphs.

Theorems 1.4 and 1.5 will be proved in the next section. Then, in Section 3, we will describe an application of Theorem 1.4 to derive a sufficient condition for the existence of a fractional independent system of representatives. In Section 4 we will point out that Theorems 1.4 and 1.5 are not specific to graphs. If we replace the independence complexes of graphs by arbitrary complexes, and suitably define domination in this general setting, the corresponding inequalities continue to hold. The case when the complexes are matroids is of particular interest. For matroids, the analogue of König's theorem is Edmonds' two matroids intersection theorem [5]. Thus, the version of Theorem 1.4 for complexes (Theorem 4.1 below) shows that the nontrivial direction of Edmonds' theorem can be extended from matroids to arbitrary complexes at the cost of fractionalizing the notion of a joint independent set.

We end the introduction by noting an interesting symmetry between two complexes associated with the same graph: the complex of independent sets and the complex of non-punctured neighborhoods. The inequality $\alpha \geqslant \gamma$ says that the number of simplices from the second complex needed to cover V is no more than the size of a largest simplex from the first complex. The basic inequality on the chromatic number of a graph, namely $\chi \leqslant \Delta + 1$, is the analogous statement with the roles of the two complexes interchanged. In Section 6 we show that the two-sided fractional versions of both of these inequalities hold true for any system of graphs. This is based on a duality between two polytopes that are naturally associated with the two complexes, which is presented in Section 5. In this duality, the two complexes play symmetric roles.

2. Proofs of Theorems 1.4 and 1.5

Although Theorem 1.4 is a special case of Theorem 1.5, we find it instructive to present its proof separately.

Proof of Theorem 1.4. Let $k = \gamma_\cup(G_1, G_2)$, and assume for the sake of contradiction that $\alpha^*_\cap(G_1, G_2) < k$. Then the convex hull of the characteristic vectors of independent sets of size k in G_1 is disjoint from the corresponding convex hull for G_2. By the separation theorem, there exists a weight function $\vec{w}$ on the common vertex set V such that $\vec{w} \cdot \chi_{I_1} > \vec{w} \cdot \chi_{I_2}$ for any two independent sets I_1, I_2 of size k in G_1, G_2, respectively. Let $<_1$ be an ordering of V so that $\vec{w}$ is non-decreasing, and let $<_2$ be the reverse ordering.

Now, let us choose an independent set $I = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ of size k in G_1 as follows. Let i_1 be the first index in the ordering $<_1$. Let i_2 be the smallest index in that ordering so that $v_{i_2} \notin N_{G_1}(\{v_{i_1}\})$. Let i_3 be the smallest index so that $v_{i_3} \notin N_{G_1}(\{v_{i_1}, v_{i_2}\})$, and so on. Note that we can carry out this process up to v_{i_k}, because a set of fewer than k vertices cannot dominate V in G_1, as such a set would imply that $\gamma_\cup(G_1, G_2) < k$. Let us also choose an independent set $J = \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\}$ of size k in G_2, by carrying out a similar process with respect to the ordering $<_2$.

As $\vec{w} \cdot \chi_I > \vec{w} \cdot \chi_J$, there must exist $1 \leqslant \ell \leqslant k$ so that $w(v_{i_\ell}) > w(v_{j_{k-\ell+1}})$. The set $N_{G_1}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_{\ell-1}}\}) \cup N_{G_2}(\{v_{j_1}, v_{j_2}, \ldots, v_{j_{k-\ell}}\})$ cannot be all of V, because $k - 1 < \gamma_\cup(G_1, G_2)$. Hence there must exist a vertex x that is neither in $N_{G_1}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_{\ell-1}}\})$ nor in $N_{G_2}(\{v_{j_1}, v_{j_2}, \ldots, v_{j_{k-\ell}}\})$. By construction, such a vertex x has to satisfy $w(v_{i_\ell}) \leqslant w(x) \leqslant w(v_{j_{k-\ell+1}})$, which contradicts the choice of ℓ. □

In preparation for the proof of Theorem 1.5 we need two lemmas. The first of them is a variant of the following fact: Given m compact and convex subsets of $\mathbb{R}^n$ with an empty intersection, one can enlarge each of them to a half-space so that the m half-spaces still have an empty intersection.

This fact is a generalization of the separation theorem from two sets to m sets, and should be well known. However, as we are not aware of a reference, and because we need a variant with hyperplanes through the origin, we provide the proof here. We remark that we could establish a stronger version with open half-spaces (i.e., strict inequality in (1) below). But we would not gain anything by doing that, because our application of the lemma involves a limiting process in which only the weak inequality is preserved.

Lemma 2.1. *Let $C_1, C_2, \dots, C_m$ (with $m \geqslant 2$) be compact and convex subsets of the set $\{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \mathbf{1} = c\}$ for some constant $c \neq 0$, and assume that $\bigcap_{i=1}^m C_i = \emptyset$. Then there exists an $m \times n$ real matrix $W = (w_{ij})$ satisfying the following three conditions*:

$$\sum_{j=1}^n w_{ij} x_j \geqslant 0 \quad \text{for every } i = 1, 2, \dots, m \text{ and every } \vec{x} = (x_1, x_2, \dots, x_n) \in C_i. \tag{1}$$

$$\sum_{i=1}^m w_{ij} = 0 \quad \text{for every } j = 1, 2, \dots, n. \tag{2}$$

$$\max\bigl\{|w_{ij}| \bigm| i = 1, 2, \dots, m,\ j = 1, 2, \dots, n\bigr\} = 1. \tag{3}$$

Proof. For $\vec{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $d \in \mathbb{R}$, we denote by $H_{\vec{y},d}$ the hyperplane $\{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{y} = d\}$, and by $H^+_{\vec{y},d}$ the closed half-space $\{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{y} \geqslant d\}$.

In the first part of the proof, we construct an auxiliary $m \times n$ real matrix $A = (a_{ij})$ that will later become, upon some normalizations, the required matrix W. We choose the rows $\vec{a}_i = (a_{i1}, a_{i2}, \dots, a_{in})$ of A one-by-one, as follows. Since the sets C_1 and $\bigcap_{i=2}^m C_i$ are disjoint, and both are compact and convex, they can be separated by a hyperplane. Moreover, since both of them are contained in $H_{\mathbf{1},c}$ and $c \neq 0$, we are free to choose the separating hyperplane so that it will pass through the origin. So, we choose $\vec{a}_1 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ so that $C_1 \subseteq H^+_{\vec{a}_1,0}$ and $H^+_{\vec{a}_1,0} \cap (\bigcap_{i=2}^m C_i) = \emptyset$. Next, we apply a similar argument to the sets C_2 and $H^+_{\vec{a}_1,0} \cap (\bigcap_{i=3}^m C_i)$ and choose $\vec{a}_2 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ so that $C_2 \subseteq H^+_{\vec{a}_2,0}$ and $H^+_{\vec{a}_1,0} \cap H^+_{\vec{a}_2,0} \cap (\bigcap_{i=3}^m C_i) = \emptyset$. After $m-1$ iterations of this argument, we obtain $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{m-1} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ so that $C_i \subseteq H^+_{\vec{a}_i,0}$, $i = 1, 2, \dots, m-1$, and $(\bigcap_{i=1}^{m-1} H^+_{\vec{a}_i,0}) \cap C_m = \emptyset$. Now, the sets C_m and $\bigcap_{i=1}^{m-1} H^+_{\vec{a}_i,0}$ are disjoint, both are closed and convex, and one of them is bounded. Hence they can be separated by a hyperplane, and so we can choose $\vec{a}_m \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $d \in \mathbb{R}$ so that $C_m \subseteq H^+_{\vec{a}_m,d}$ and $(\bigcap_{i=1}^{m-1} H^+_{\vec{a}_i,0}) \cap H^+_{\vec{a}_m,d} = \emptyset$. Note that the latter implies that $d > 0$ (otherwise $\mathbf{0}$ would lie in the intersection), and therefore $C_m \subseteq H^+_{\vec{a}_m,0}$. It follows that (1) is satisfied by the matrix $A = (a_{ij})$.

We observe that the system of inequalities $\vec{a}_i \cdot \vec{x} \geqslant 0$, $i = 1, 2, \dots, m-1$, implies the inequality $-\vec{a}_m \cdot \vec{x} \geqslant 0$, because if $\vec{x}$ is a solution of the system that satisfies $-\vec{a}_m \cdot \vec{x} < 0$ then a suitable multiple of $\vec{x}$ lies in $(\bigcap_{i=1}^{m-1} H^+_{\vec{a}_i,0}) \cap H^+_{\vec{a}_m,d}$. It follows that $-\vec{a}_m$ lies in the convex cone generated by $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{m-1}$. So there exist non-negative coefficients $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$, not all of them zero, such that $-\vec{a}_m = \sum_{i=1}^{m-1} \lambda_i \vec{a}_i$. We can now define the $m \times n$ matrix $B = (b_{ij})$ by $b_{ij} = \lambda_i a_{ij}$ for $i = 1, 2, \dots, m-1$ and $b_{mj} = a_{mj}$, and note that (1) and (2) are satisfied by the matrix $B = (b_{ij})$. Since every row of A had a non-zero entry, B is not the zero matrix. Multiplying B by a suitable positive factor, we obtain a matrix $W = (w_{ij})$ satisfying all three conditions in the lemma. $\square$

The second lemma is a tool used by Lovász [9] in his proof that $\tau \leqslant \frac{m}{2}\tau^*$ for m-partite m-graphs. For the reader's convenience, and because we need an additional property (condition (6) below), we reproduce the proof here.

Lemma 2.2. *For all integers $m \geqslant 2$ and $k \geqslant 1$ there exists an $m \times k$ matrix $L = (\ell_{ij})$ satisfying the following three conditions*:

$$\text{Every row of } L \text{ is a permutation of } \{0, 1, \ldots, k-1\}. \tag{4}$$

$$\text{The sum of every column of } L \text{ is either } \left\lfloor \frac{m(k-1)}{2} \right\rfloor \text{ or } \left\lceil \frac{m(k-1)}{2} \right\rceil. \tag{5}$$

$$\text{Every } 0 \leqslant p \leqslant k-1 \text{ appears at least once in a column with sum } \left\lfloor \frac{m(k-1)}{2} \right\rfloor. \tag{6}$$

Proof. Note that every $m \times k$ matrix whose rows are permutations of $\{0, 1, \ldots, k-1\}$ has average column sum $\frac{m(k-1)}{2}$. Thus (5) requires that if $\frac{m(k-1)}{2}$ is an integer then all column sums should be equal, and otherwise they should differ by at most 1. Condition (6) is an extra requirement only when $\frac{m(k-1)}{2}$ is not an integer.

It suffices to construct suitable matrices for $m = 2$ and $m = 3$, because for arbitrary m we can use then, if m is even, $\frac{m}{2}$ blocks of order $2 \times k$, and if m is odd, $\frac{m-3}{2}$ blocks of order $2 \times k$ and one block of order $3 \times k$. The construction for $m = 2$ is trivial:

$$\begin{pmatrix} 0 & 1 & \cdots & k-2 & k-1 \\ k-1 & k-2 & \cdots & 1 & 0 \end{pmatrix}.$$

A valid construction for $m = 3$ and k even, say $k = 2q$, is

$$\begin{pmatrix} 0 & 1 & \cdots & q-2 & q-1 & q & q+1 & \cdots & 2q-2 & 2q-1 \\ 2q-1 & 2q-3 & \cdots & 3 & 1 & 2q-2 & 2q-4 & \cdots & 2 & 0 \\ q & q+1 & \cdots & 2q-2 & 2q-1 & 0 & 1 & \cdots & q-2 & q-1 \end{pmatrix}.$$

It is easy to check that each of the first q columns adds up to $3q - 1$, each of the last q columns adds up to $3q - 2$, and the latter contain at least one appearance of each $0 \leqslant p \leqslant 2q - 1$. A slight variation of this construction works for $m = 3$ and k odd. (We omit the details for this case, in view of the fact that in the application below k may be assumed to be even.) □

Proof of Theorem 1.5. We treat here the case $m > 2$, having already handled the case $m = 2$ in proving Theorem 1.4. We assume, for the sake of contradiction, that we have a system $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ of m graphs on a common vertex set V so that $\alpha^*_\cap(\mathcal{G}) \leqslant \frac{2}{m}\gamma_\cup(\mathcal{G})$. Note that if we consider an r-fold replication of our system, namely we take r disjoint copies of V and look at the system $r\mathcal{G} = (rG_1, rG_2, \ldots, rG_m)$, we get $\alpha^*_\cap(r\mathcal{G}) = r\alpha^*_\cap(\mathcal{G})$ and $\gamma_\cup(r\mathcal{G}) = r\gamma_\cup(\mathcal{G})$. Hence, if $\mathcal{G}$ is a counterexample then so is $r\mathcal{G}$. Choosing $r = m$, we see that $\frac{2}{m}\gamma_\cup(m\mathcal{G})$ is an integer. Hence, from now on we will assume that our system $\mathcal{G}$ satisfies $\alpha^*_\cap(\mathcal{G}) \leqslant \frac{2}{m}\gamma_\cup(\mathcal{G}) = k$ for some positive integer k. We also let $v_1, v_2, \ldots, v_n$ be an enumeration of the vertices of the system $\mathcal{G}$.

Let $\varepsilon > 0$. The fact that $\alpha^*_\cap(\mathcal{G}) < k + \varepsilon$ implies that the sets

$$C^\varepsilon_i = \Omega\big(\mathcal{I}(G_i)\big) \cap \big\{\vec{x} \in \mathbb{R}^n \;\big|\; \vec{x} \cdot \mathbf{1} = k + \varepsilon\big\}, \quad i = 1, 2, \ldots, m,$$

have an empty intersection. Hence there exists an $m \times n$ matrix $W^\varepsilon = (w^\varepsilon_{ij})$ satisfying the conditions of Lemma 2.1. Now, consider a sequence $\varepsilon_\ell \to 0$ and a corresponding sequence of matrices W^{ε_ℓ}. Condition (3) guarantees that the sequence W^{ε_ℓ} has a subsequence that converges to a matrix $W = (w_{ij})$ which is not the zero matrix. This matrix satisfies (2) and

$$\sum_{v_j \in I} w_{ij} \geqslant 0 \quad \text{for every } i = 1, 2, \ldots, m \text{ and every } I \in \mathcal{I}(G_i) \text{ such that } |I| = k. \tag{7}$$

To verify (7), suppose I is an independent set in G_i of size k. Since $k < \gamma_\cup(\mathcal{G}) \leqslant \gamma(G_i) \leqslant \alpha(G_i)$ there exists an independent set J in G_i of size $k + 1$. For $0 < \varepsilon \leqslant 1$, the vector $(1 - \varepsilon)\chi_I + \varepsilon\chi_J$ lies in C_i^ε, and hence satisfies the weak inequality (1) with respect to $(w_{i1}^\varepsilon, w_{i2}^\varepsilon, \ldots, w_{in}^\varepsilon)$. As $\varepsilon \to 0$ we get the required inequality.

It will be convenient to view each row of the matrix W as a weight function on the vertex set V. The weight function represented by the ith row will be denoted by $\vec{w}_i$, and its value on vertex v_j will be written in the form $w_i(v_j)$. For each $i = 1, 2, \ldots, m$, let $<_i$ be an ordering of V so that $\vec{w}_i$ is non-decreasing. For every i separately, we choose an independent set $I_i = \{v_{i,0}, v_{i,1}, \ldots, v_{i,k-1}\}$ in G_i of size k, as follows. Let $v_{i,0}$ be the first vertex in the ordering $<_i$. Let $v_{i,1}$ be the first element of $V \setminus N_{G_i}(\{v_{i,0}\})$ in that ordering. Let $v_{i,2}$ be the first element of $V \setminus N_{G_i}(\{v_{i,0}, v_{i,1}\})$, and so on. We can carry out this process because $k < \gamma(G_i)$. For $r = 0, 1, \ldots, k - 1$ we let $I_{i,r} = \{v_{i,0}, v_{i,1}, \ldots, v_{i,r-1}\}$ consist of the first r elements of I_i. By construction, if vertex x is not dominated in G_i by $I_{i,r}$, then $w_i(x) \geqslant w_i(v_{i,r})$.

We consider now an $m \times k$ matrix $L = (\ell_{ij})$ satisfying the conditions of Lemma 2.2. Fixing a column j, we look at the sets $I_{i,\ell_{ij}}$, $i = 1, 2, \ldots, m$, and at their neighborhoods $N_{G_i}(I_{i,\ell_{ij}})$ in the respective graphs. The total size of the sets $I_{i,\ell_{ij}}$ is

$$\sum_{i=1}^{m} \ell_{ij} \leqslant \left\lceil \frac{m(k-1)}{2} \right\rceil = \left\lceil \gamma_\cup(\mathcal{G}) - \frac{m}{2} \right\rceil \leqslant \gamma_\cup(\mathcal{G}) - 1 \tag{8}$$

and therefore their neighborhoods cannot cover V. Hence there exists a vertex x which belongs to none of these neighborhoods, and therefore satisfies $w_i(x) \geqslant w_i(v_{i,\ell_{ij}})$ for $i = 1, 2, \ldots, m$. By (2) the left-hand sides of these inequalities add up to zero, and we conclude that

$$\sum_{i=1}^{m} w_i(v_{i,\ell_{ij}}) \leqslant 0 \quad \text{for every } j = 1, 2, \ldots, k. \tag{9}$$

When we sum these k inequalities, every term $w_i(v_{i,p})$ for $i = 1, 2, \ldots, m$ and $p = 0, 1, \ldots, k - 1$, appears exactly once, because each row of L is a permutation of $\{0, 1, \ldots, k - 1\}$. It follows that $\sum_{i=1}^{m} \sum_{p=0}^{k-1} w_i(v_{i,p}) \leqslant 0$. On the other hand, by (7) we have

$$\sum_{p=0}^{k-1} w_i(v_{i,p}) \geqslant 0 \quad \text{for every } i = 1, 2, \ldots, m. \tag{10}$$

We conclude that the double sum is actually zero, and both (9) and (10) hold as equalities throughout.

We now claim that

$$w_i(v_{i,p}) = w_i(v_{i,p+1}) \quad \text{for every } i = 1, 2, \ldots, m \text{ and } p = 0, 1, \ldots, k - 2. \tag{11}$$

Indeed, suppose that $w_i(v_{i,p}) < w_i(v_{i,p+1})$ for some i and p. By (6) p appears in some column j of the matrix L having sum $\lfloor \frac{m(k-1)}{2} \rfloor$. We may assume that $p = \ell_{ij}$, otherwise we can permute the rows of L to achieve this, and redo the above argument with the row-permuted matrix. Now we reconsider the calculation in (8), and observe that for this particular j the total size is at most

$\gamma_\cup(\mathcal{G}) - 2$ (since the ceiling is now a floor and $m \geqslant 3$). Hence the argument above remains valid if we increase the set $I_{i,p}$ (for this i) to become $I_{i,p+1}$, and so the weak inequality (9) continues to hold for this j if we replace the term $w_i(v_{i,p})$ by $w_i(v_{i,p+1})$. This is a contradiction, as by our assumptions (9) held as an equality, and the replacement strictly increased its left-hand side.

It follows from (11) and the equality in (10) that $w_i(v_{i,p}) = 0$ for every $i = 1, 2, \dots, m$ and $p = 0, 1, \dots, k-1$. Hence we have $w_i(v_j) \geqslant w_i(v_{i,0}) = 0$ for every $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. By (2) this implies that W is the zero matrix, a contradiction. □

3. Fractional independent systems of representatives

We present here an application of Theorem 1.4. Given disjoint sets $V_1, V_2, \dots, V_k$ and a graph G on $V = \bigcup_{i=1}^k V_i$, an *independent system of representatives* (ISR) is a set which is independent in G and contains a vertex from each set V_i. It is known [7] that a sufficient condition for the existence of an ISR is that $\gamma(G[\bigcup_{j\in J} V_j]) \geqslant 2|J| - 1$ for every subset J of $\{1, 2, \dots, k\}$. Another sufficient condition [2] is that $\gamma^i(G[\bigcup_{j\in J} V_j]) \geqslant |J|$ for every $J \subseteq \{1, 2, \dots, k\}$, where $\gamma^i(H)$ is the maximum, over all independent sets S of vertices in the graph H, of the minimal size of a set that dominates S. Note that $\gamma^i(H) \leqslant \gamma(H)$ and the inequality may be strict, for example when H is a 4-cycle. Hence the two sufficient conditions are not logically comparable. Their natural common weakening, requiring that $\gamma(G[\bigcup_{j\in J} V_j]) \geqslant |J|$ for every $J \subseteq \{1, 2, \dots, k\}$, does not suffice for the existence of an ISR. But we will show that it does suffice for the existence of a fractional version of it.

A *fractional ISR* in the above setting is a non-negative function f on V belonging to $\Omega(\mathcal{I}(G))$, such that $\sum_{v\in V_i} f(v) \geqslant 1$ for $i = 1, 2, \dots, k$. An application of Theorem 1.4 with $G_1 = G$ and G_2 the partition graph with cliques $V_1, V_2, \dots, V_k$ shows that a fractional ISR is guaranteed to exist if $\gamma_\cup(G_1, G_2) = k$. This, in turn, will hold true if for every $J \subseteq \{1, 2, \dots, k\}$, the set $\bigcup_{j\in J} V_j$ cannot be covered by fewer than $|J|$ neighborhoods in G. This sufficient condition is somewhat stronger than promised above, because in $\gamma(G[\bigcup_{j\in J} V_j])$ we consider only neighborhoods of vertices that are themselves in $\bigcup_{j\in J} V_j$. We do obtain the sufficiency of the weaker version by a direct proof below.

Theorem 3.1. *Let G be a graph, and $V = \bigcup_{i=1}^k V_i$ be a partition of its vertex set. If $\gamma(G[\bigcup_{j\in J} V_j]) \geqslant |J|$ for every $J \subseteq \{1, 2, \dots, k\}$ then there exists a fractional ISR.*

Proof. Consider the problem of minimizing $\sum_I x_I$ over all assignments of non-negative weights x_I to the independent sets of G such that $\sum_I |I \cap V_j| x_I \geqslant 1$ for $j = 1, 2, \dots, k$. Clearly, a fractional ISR exists if and only if the value of this linear program is at most 1. By linear programming duality, it suffices to show that under the theorem's condition the maximum of $\sum_{j=1}^k y_j$, subject to $y_j \geqslant 0$ for $j = 1, 2, \dots, k$ and $\sum_{j=1}^k |I \cap V_j|\, y_j \leqslant 1$ for every independent set I of G, is at most 1.

Suppose that $(y_1, y_2, \dots, y_k)$ satisfies the constraints and, without loss of generality, $y_1 \geqslant y_2 \geqslant \dots \geqslant y_k$. The theorem's condition allows us to construct an independent set $I = \{v_1, v_2, \dots, v_k\}$ as follows. Let v_1 be an arbitrary vertex in V_1, let v_2 be a vertex in $V_1 \cup V_2$ not adjacent to v_1, let v_3 be a vertex in $V_1 \cup V_2 \cup V_3$ not adjacent to v_1 or v_2, and so on. By assumption we have $\sum_{j=1}^k |I \cap V_j| y_j \leqslant 1$, and by construction we have $\sum_{j=1}^{\ell} |I \cap V_j| \geqslant \ell$ for every $\ell = 1, 2, \dots, k$. As $y_1 \geqslant y_2 \geqslant \dots \geqslant y_k$, it follows by a standard majorization argument that $\sum_{j=1}^k y_j \leqslant 1$, as required. □

4. General complexes

In fact, nothing that we have done so far is particular to graphs—all notions and results can be extended to general complexes. For a simplex σ in a complex $\mathcal{C}$ on V we write $N_{\mathcal{C}}(\sigma) = \sigma \cup \{y \in V \setminus \sigma \mid \sigma \cup \{y\} \notin \mathcal{C}\}$. For a subset X of V, we denote by $\mathcal{C} \restriction X$ the complex consisting of those simplices in $\mathcal{C}$ that are contained in X. The *span* $sp_{\mathcal{C}}(X)$ of a set X is $\bigcup\{N_{\mathcal{C}}(\sigma) \mid \sigma \in \mathcal{C} \restriction X\}$. Note that when $\mathcal{C}$ is a matroid, i.e., the simplices are the independent sets of a matroid, the span definition is the usual matroidal span. For a complex $\mathcal{C}$ on V write $\gamma(\mathcal{C}) = \min\{|X| \mid sp_{\mathcal{C}}(X) = V\}$. For a system of complexes $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ write $\gamma_{\cup}(\mathcal{L}) = \min\{\sum_{i=1}^m |X_i| \mid \bigcup_{i=1}^m sp_{\mathcal{C}_i}(X_i) = V\}$.

The same proofs as those of Theorems 1.4 and 1.5 yield:

Theorem 4.1. *For any two complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ on the same ground set we have*

$$\mu^*_{\cap}(\mathcal{C}_1, \mathcal{C}_2) \geqslant \gamma_{\cup}(\mathcal{C}_1, \mathcal{C}_2).$$

Theorem 4.2. *For a system $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m)$ of $m \geqslant 2$ complexes on the same ground set we have*

$$\mu^*_{\cap}(\mathcal{L}) \geqslant \frac{2}{m}\gamma_{\cup}(\mathcal{L}),$$

with strict inequality for $m > 2$.

We note that in the case of two matroids, Theorem 4.1 holds true even without fractionalizing $\mu_{\cap}$; this is Edmonds' two matroids intersection theorem [5]. In this case, the weak inequality is actually an equality.

5. A duality between the independence complex and the non-punctured neighborhood complex

For a polytope $P \subseteq \mathbb{R}^n_+$ we denote by $\bar{P}$ the set $\{\vec{x} \in \mathbb{R}^n_+ \mid \vec{x} \cdot \vec{y} \leqslant 1 \ \forall \vec{y} \in P\}$. The polytope $\bar{P}$ is called the "anti-blocker" of P (see [6]). A polytope $P \subseteq \mathbb{R}^n_+$ is said to be *closed down* if $\vec{x} \in P, \vec{y} \in \mathbb{R}^n_+, \vec{y} \leqslant \vec{x}$ imply that $\vec{y} \in P$. Using the separation theorem it is easy to show (see, e.g., [6]):

Lemma 5.1. *If $P \subseteq \mathbb{R}^n_+$ is a closed down polytope then $\bar{\bar{P}} = P$.*

Since clearly $P \subseteq Q$ implies $\bar{Q} \subseteq \bar{P}$, Lemma 5.1 implies:

Lemma 5.2. *Let P and Q be closed down polytopes in $\mathbb{R}^n_+$. Then $\bar{Q} \subseteq P$ if and only if $\bar{P} \subseteq Q$.*

If the relation $\bar{Q} \subseteq P$ holds, we say that P and Q are *barring*.

Theorem 5.3. *For any graph G the polytopes $\Omega(\mathcal{I}(G))$ and $\Omega(\mathcal{N}(G))$ are barring.*

Proof. Write $P = \Omega(\mathcal{I}(G))$, $Q = \Omega(\mathcal{N}(G))$. Assuming the negation of the theorem, there exists $\vec{y} \in \bar{P} \setminus Q$. By the separation theorem there exists then a non-negative weight function $\vec{w}$ on

V such that $\vec{w} \cdot \vec{y} > 1 > \vec{w} \cdot \chi_N$ for every $N \in \mathcal{N}(G)$. Order V in such a way that $\vec{y}$ is non-increasing. Choose now an independent set I in G "greedily," which is formally done as follows. Let $i_1 = 1$. Let i_2 be the first index such that $v_{i_2} \notin N(\{v_{i_1}\})$, let i_3 be the first index such that $v_{i_3} \notin N(\{v_{i_1}, v_{i_2}\})$, and so on. Finally, let $I = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$, where v_{i_k} is the last vertex chosen in this process. Since $\vec{y} \in \bar{P}$ we have $\vec{y} \cdot \chi_I \leqslant 1$. By breaking the summation in $\vec{w} \cdot \vec{y}$ into k parts, each over one of the sets $N(\{v_{i_j}\}) \setminus N(\{v_{i_1}, v_{i_2}, \ldots, v_{i_{j-1}}\})$, $j = 1, 2, \ldots, k$, and using the fact that $\vec{y}$ is non-increasing, we have

$$\vec{w} \cdot \vec{y} \leqslant \sum_{j=1}^{k} y(v_{i_j}) \sum \bigl\{ w(x) \bigm| x \in N\bigl(\{v_{i_j}\}\bigr) \bigr\}.$$

For each j, the second sum on the right-hand side equals $\vec{w} \cdot \chi_{N(\{v_{i_j}\})}$, which is less than 1 by our choice of $\vec{w}$. It follows that

$$\vec{w} \cdot \vec{y} < \vec{y} \cdot \chi_I \leqslant 1,$$

which yields the desired contradiction. □

Theorem 5.3 generalizes from graphs to arbitrary complexes, but we need to be somewhat careful. In this generalization, the role of $\Omega(\mathcal{N}(G))$ will be played by the convex hull not just of the characteristic vectors of sets spanned by single vertices, but also of the proportionally normalized characteristic vectors of sets spanned by multiple vertices. To be precise, we define for a complex $\mathcal{C}$ on V the *span polytope*

$$\Theta(\mathcal{C}) = \operatorname{conv}\left\{ \frac{1}{|X|} \chi_Z \;\middle|\; \emptyset \neq X \subseteq V,\ Z \subseteq sp_{\mathcal{C}}(X) \right\}.$$

Theorem 5.4. *For any complex $\mathcal{C}$ the polytopes $\Omega(\mathcal{C})$ and $\Theta(\mathcal{C})$ are barring.*

Proof. Write $P = \Omega(\mathcal{C})$, $Q = \Theta(\mathcal{C})$. Following the proof of the previous theorem, we assume the negation and obtain $\vec{y} \in \bar{P} \setminus Q$ and $\vec{w}$ on V such that $\vec{w} \cdot \vec{y} > 1 > \vec{w} \cdot \frac{1}{|X|}\chi_Z$ for every $\emptyset \neq X \subseteq V$ and $Z \subseteq sp_{\mathcal{C}}(X)$. We order V so that $\vec{y}$ is non-increasing, and choose a simplex $\sigma = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ in $\mathcal{C}$ by the process where each v_{i_j} is the first vertex not in $sp_{\mathcal{C}}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_{j-1}}\})$, as long as we can find such a vertex. We partition V into the sets N_j, $j = 1, 2, \ldots, k$, where $N_j = sp_{\mathcal{C}}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}) \setminus sp_{\mathcal{C}}(\{v_{i_1}, v_{i_2}, \ldots, v_{i_{j-1}}\})$. Denoting $\sum\{w(x) \mid x \in N_j\}$ by a_j, and observing that $y(v_{i_j})$ is the maximum value assumed by $\vec{y}$ over N_j, we have

$$\vec{w} \cdot \vec{y} \leqslant \sum_{j=1}^{k} a_j y(v_{i_j}). \tag{12}$$

By the separation condition above, we have for each $\ell = 1, 2, \ldots, k$,

$$\sum_{j=1}^{\ell} a_j = \sum \bigl\{ w(x) \bigm| x \in sp_{\mathcal{C}}\bigl(\{v_{i_1}, v_{i_2}, \ldots, v_{i_\ell}\}\bigr) \bigr\} < \ell.$$

In other words, each partial sum of $\sum_{j=1}^{k} a_j$ is smaller than the corresponding one of $\sum_{j=1}^{k} 1$. As $\vec{y}$ is non-increasing, it follows by a standard majorization argument that $\sum_{j=1}^{k} a_j y(v_{i_j}) \leqslant \sum_{j=1}^{k} y(v_{i_j})$. Returning to (12), and bearing in mind that $\vec{y} \in \bar{P}$ implies that $\vec{y} \cdot \chi_\sigma \leqslant 1$, we get

$$\vec{w} \cdot \vec{y} \leqslant \sum_{j=1}^{k} a_j y(v_{i_j}) \leqslant \sum_{j=1}^{k} y(v_{i_j}) = \vec{y} \cdot \chi_\sigma \leqslant 1$$

a contradiction. □

We remark that if $\mathcal{C}$ is a matroid then the inclusion proved above, $\overline{\Theta(\mathcal{C})} \subseteq \Omega(\mathcal{C})$, is actually an equality, because the reverse inclusion is easily seen to hold for a matroid. So we get as a corollary that $\Omega(\mathcal{C}) = \overline{\Theta(\mathcal{C})}$ when $\mathcal{C}$ is a matroid. This amounts to a characterization of the convex hull of the independent sets of a matroid, originally due to Edmonds [4].

6. Fractionalizing on both sides of the inequality

It turns out that the inequality of the form $\alpha_\cap \geqslant \gamma_\cup$ becomes true for any number of graphs (or in fact, complexes) if we take fractional versions of both sides of the inequality.

Given a system $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of complexes on V, let $\chi_\cup^*(\mathcal{L})$ be defined as $\min \sum f(\sigma)$, where the minimum is taken over all functions $f : \bigcup_{i=1}^{m} \mathcal{C}_i \to \mathbb{R}_+$ which fractionally cover V, namely $\sum\{f(\sigma) \mid v \in \sigma\} \geqslant 1$ for all $v \in V$. For a system $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ of graphs we write $\gamma_\cup^*(\mathcal{G}) = \chi_\cup^*(\mathcal{N}(G_1), \mathcal{N}(G_2), \ldots, \mathcal{N}(G_m))$.

Theorem 6.1. *For any system $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ of m graphs on the same vertex set we have*

$$\alpha_\cap^*(\mathcal{G}) \geqslant \gamma_\cup^*(\mathcal{G}).$$

Proof. The minimization problem that defines $\gamma_\cup^*(\mathcal{G})$ has a linear programming dual: it is the problem of maximizing $\vec{x} \cdot \mathbf{1}$, where $\vec{x}$ ranges over all non-negative weight functions on V satisfying $\vec{x} \cdot \chi_N \leqslant 1$ for every neighborhood N in any of the graphs $G_1, G_2, \ldots, G_m$. Therefore, it suffices to show that for any one graph G_i, a vector $\vec{x}$ that satisfies these constraints must lie in $\Omega(\mathcal{I}(G_i))$. This follows from Theorem 5.3. □

The most basic inequality on the chromatic number of a graph is $\chi(G) \leqslant \mu(\mathcal{N}(G))$ (the right-hand side being more familiar under the notation $\Delta(G) + 1$). The following theorem relates to this inequality in the same way that Theorem 6.1 relates to the inequality $\alpha \geqslant \gamma$.

Theorem 6.2. *For any system $\mathcal{G} = (G_1, G_2, \ldots, G_m)$ of m graphs on the same vertex set we have*

$$\chi_\cup^*\big(\mathcal{I}(G_1), \mathcal{I}(G_2), \ldots, \mathcal{I}(G_m)\big) \leqslant \mu_\cap^*\big(\mathcal{N}(G_1), \mathcal{N}(G_2), \ldots, \mathcal{N}(G_m)\big).$$

Proof. The minimization problem that defines $\chi_\cup^*(\mathcal{I}(G_1), \mathcal{I}(G_2), \ldots, \mathcal{I}(G_m))$ has a linear programming dual: it is the problem of maximizing $\vec{x} \cdot \mathbf{1}$, where $\vec{x}$ ranges over all non-negative weight functions on V satisfying $\vec{x} \cdot \chi_I \leqslant 1$ for every independent set I in any of the graphs $G_1, G_2, \ldots, G_m$. Therefore, it suffices to show that for any one graph G_i, a vector $\vec{x}$ that satisfies these constraints must lie in $\Omega(\mathcal{N}(G_i))$. This follows from Theorem 5.3. □

Note the similarity of the last two proofs, and the fact that we used in them two equivalent interpretations of Theorem 5.3: $\overline{\Omega(\mathcal{N}(G))} \subseteq \Omega(\mathcal{I}(G))$ for the former, $\overline{\Omega(\mathcal{I}(G))} \subseteq \Omega(\mathcal{N}(G))$ for the latter.

We remark also that both theorems can be generalized to systems $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of complexes on the same ground set V. Let $\gamma_{\cup}^*(\mathcal{L})$ be defined as $\min \sum f(i, X)$, where the minimum is taken over all non-negative functions f defined on pairs (i, X) with $i \in \{1, 2, \ldots, m\}$ and $\emptyset \neq X \subseteq V$ that satisfy $\sum f(i, X) \frac{1}{|X|} \chi_{sp_{\mathcal{C}_i}(X)} \geqslant \mathbf{1}$. The analogue of Theorem 6.1 is

Theorem 6.3. *For any system $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of m complexes on the same ground set we have*

$$\mu_{\cap}^*(\mathcal{L}) \geqslant \gamma_{\cup}^*(\mathcal{L}).$$

The analogue of Theorem 6.2 is stated next, using the definition of the span polytope $\Theta(\mathcal{C})$ given in the previous section.

Theorem 6.4. *For any system $\mathcal{L} = (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of m complexes on the same ground set we have*

$$\chi_{\cup}^*(\mathcal{L}) \leqslant \max\left\{ \vec{x} \cdot \mathbf{1} \;\middle|\; \vec{x} \in \bigcap_{i=1}^{m} \Theta(\mathcal{C}_i) \right\}.$$

The proofs of the last two theorems are similar to the above, using Theorem 5.4 instead of Theorem 5.3. In the case when $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ are matroids, it can be shown that the weak inequalities in Theorems 6.3 and 6.4 actually hold as equalities.

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