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Matching of like rank and the size of the core in the marriage problem ☆

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ABSTRACT

When men and women are objectively ranked in a marriage problem, say by beauty, then pairing individuals of equal rank is the only stable matching. We generalize this observation by providing bounds on the size of the rank gap between mates in a stable matching in terms of the size of the ranking sets. Using a metric on the set of matchings, we provide bounds on the diameter of the core – the set of stable matchings – in terms of the size of the ranking sets and in terms of the size of the rank gap. We conclude that when the set of rankings is small, so are the core and the rank gap in stable matchings. We construct examples showing that our bounds are essentially tight, and that certain natural variants of the bounds fail to hold.

1. Introduction

1.1. Matching of likes

When considering the dazzling world of stardom and glamor we are not at all surprised to see Angelina Jolie and Brad Pitt as a couple. Both are highly ranked in this world, and their match seems natural. We would be bewildered, on the other hand, to see Jolie matched up with another man of this world whose physical appearance ranks much lower than hers. Such a man, so we expect, would be naturally matched with a woman ranked like him.

Those who are not familiar with the world of entertainment, may find it easier to relate to a similar mating of likes in the academic arena. Highly ranked scholars are affiliated, more often than not, with top-tier universities, while those who are academically less attractive are affiliated with lesser universities.

The main purpose of this paper is to explain the phenomenon of matching of likes, also known as assortative matching, within the framework of the marriage problem introduced by Gale and Shapley (1962). We recall that the solution concept for that problem is the core, i.e., the set of stable matchings.

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1.2. The case of a universal ranking

Our benchmark case is that of *universal* rankings, namely: all men rank women in the same way and all women rank men in the same way. It is well known that in this case the only stable matching matches the highest ranked man to the highest ranked woman, the second highest ranked man to the second highest ranked woman, and so on. Thus, in this very special case, matched individuals have equal ranks.

In this paper we show that the observation above is robust. That is, if rankings are not necessarily universal, but are similar to each other, then in any stable matching, matched individuals have similar ranks. Additionally, if rankings are similar to each other then, though different stable matchings may exist, they must be close to each other. To make these statements precise, we will introduce below quantitative measures of how similar rankings are (the *size of the ranking sets*), how similar the ranks of mates are (the *rank gaps*), and how close to each other stable matchings are (the *size of the core*).

1.3. Correlated rankings

The results of this paper (stated informally in the previous paragraph) suggest the similarity of rankings by different individuals as a possible explanation of two real-life phenomena: matching of likes and smallness of the core.

To what extent is the assumption of similar rankings by different individuals realistic? Obviously, the assumption of a universal ranking is too strong. We can hardly expect a unanimous, universal ranking in anything that involves human beings. However, if there is some objective component in the rankings, then individual rankings will be positively correlated; the more significant the objective component is, the more similar the rankings will tend to be. This may often be the case. Beauty, for example, is indeed in the eyes of the beholder. Nevertheless, in a given culture there is a great deal of agreement in the judgement of beauty, and rankings of beauty are positively correlated. Similarly, scholars may differ on the ranking of universities, but in all rankings Harvard is among the top, say, ten universities. Below, we use the bound on the range of ranks of each individual in a set of rankings as the basis for a measure of the size of that set of rankings.

1.4. Rank gaps and the size of the core

The notion of rank gap is meant to give a quantitative answer to the question how close a given matching is to being assortative. A quantitative way to express the fact that a matching is assortative (in the case of universal ranking) is to say that for every matched pair, the proportion of men who are better than the husband equals the proportion of women who are better than the wife. This suggests that a matching is close to being assortative if the proportions of preferred individuals to each of two mates are close to each other. With this in mind, we look at each pair in the matching, and define the rank gap as the absolute value of the difference between the woman's rank in the man's ranking and the man's rank in the woman's ranking. Then we consider the maximum, or the average, of these rank gaps over all matched pairs as our measures of closeness of the matching to being assortative.

Note that in general, when rankings are not universal, the rank of an individual (and hence the proportion of individuals who are better than him/her) is not objectively defined. Thus, in order to compare these proportions between mates, we need to use subjective ranks, and the most natural choice is to look at the ranks of the two mates in the eyes of each other. Since we are interested here in the case where rankings by different individuals are similar, this choice of ranks may be thought of as an approximation of the nearly universal rankings.

The relevant measure of the size of the core is not the *number* of stable matchings, but the maximal distance between them. From the point of view of a woman, the core is small if the ranks of the men she is matched to in the best and worst stable matchings are close. The core is small for the women if it is small for them on average. A similar measure of the size of the core can be defined for men.

1.5. The main results

Equipped with precise definitions of the above mentioned measures, we give here bounds on the rank gaps in stable matchings and on the size of the core, in terms of the size of the sets of rankings. In particular it follows that when these sets are small, that is, when rankings are highly correlated, then the rank gaps are small and the core is also small. We show, moreover, that the size of the core has a bound in terms of the rank gaps of the two extremal stable matchings.

In the last section, we construct a few examples showing that these bounds are (at least essentially) tight. We also provide counterexamples to a number of (seemingly natural) variants of our bounds. In particular, it turns out that in order for the core to be small for the women, it is not enough that the men hold highly correlated rankings. We also show that an alternative way to measure the size of a set of rankings, say of the women by the men, which considers it small if for any two men, the average woman is ranked by them similarly, cannot provide bounds on the size of the core and the rank gaps. Finally, we show that our bounds on the size of the core cannot be strengthened to assert that the core is small for every individual (not just on average).

1.6. Related work

It is well known that for general rankings, the rank gaps in stable matchings and the core may be large. In fact, Pittel (1989) proved that if n men and n women each rank the members of the opposite gender independently and uniformly at random, then in the man-optimal stable matching, with high probability as $n \to \infty$, the average man ranks his wife in rank $\sim \ln n$, whereas the average woman ranks her husband in rank $\sim \frac{n}{\ln n}$. Thus, the average rank gap is at least $\frac{n}{\ln n} - \ln n$. Moreover, using the analogous result for the woman-optimal stable matching, it follows that the rank difference between one's partners in the two extremal stable matchings (i.e., the size of the core) is $\frac{n}{\ln n} - \ln n$. This shows that significant rank gaps and a large core are not only possible, but in fact typical, for large marriage markets with random rankings.

Nevertheless, using large data sets from the National Resident Matching Program, Roth and Peranson (1999) observed empirically that the core of these markets tends to be small, and wrote:

> "One factor that strongly influences the size of the set of stable matchings (which coincides with the core in this simple model) is the correlation of preferences among programs and among applicants. When preferences are highly correlated (i.e., when similar programs tend to agree which are the most desirable applicants, and applicants tend to agree which are the most desirable programs), the set of stable matchings is small."

The theoretical result here, bounding the size of the core in terms of the size of the ranking sets, is a formalization of this observation.

Following Roth and Peranson (1999), there is a growing literature on stable matchings in large random markets. Most of these papers obtain results showing that, with high probability as the size of the market tends to infinity, the size of the core vanishes. But, in view of Pittel's result cited above, each of these results is driven by one or more special assumptions which distinguish the model studied from Pittel's plain random model. In particular, Immorlica and Mahdian (2005) and Kojima and Pathak (2009) assume that the members of one side of the market rank only a small (vanishing) fraction of the members of the other side; the unranked individuals are considered unacceptable to them. Lee (2011) introduces cardinal utilities which are chosen at random within a fixed interval, and measures the size of the core in these cardinal utility units.[1] Azevedo and Leshno (2012) consider a many-to-one setting with a constant number of schools on one side, and an increasing number of students, modeling the limit as having a continuum of students. Bodoh-Creed (2013) also studies a continuum of agents, using a type-space to describe their characteristics. Ashlagi et al. (2013) consider finite one-to-one marriage markets with unequal numbers of men and women.

The above-mentioned results in this literature may be interpreted as pointing to their respective special modeling assumptions as reasons behind the empirical finding that cores of large markets tend to be small. Our result points to a different reason – the correlation of rankings – which was also postulated by Roth and Peranson (1999). There are several other features that distinguish our approach from this recent literature. We deal not only with the size of the core, but also with the rank gaps. Our results apply to any given market with known bounds on the size of the ranking sets, rather than to most markets under some specific probabilistic model of sampling them. Moreover, our results apply to markets of any size, rather than being asymptotic for large markets.

Eeckhout (2000) and Clark (2006) gave conditions on the rankings that are sufficient for the uniqueness of stable matchings. However, they did not investigate conditions under which the set of stable matchings, though not necessarily a singleton, must be small.

Caldarelli and Capocci (2001) and Boudreau and Knoblauch (2010) studied correlation of rankings via statistical simulation. They introduce an objective trait of agents measured numerically, and assigning the value of this trait to individuals by random variables, they generate correlated and intercorrelated rankings. The simulations are restricted to the optimal matchings obtained by the deferred acceptance algorithm. Their main interest is in gender satisfaction, which is the sum of the ranks of the women by their mates in the optimal matchings.

2. Preliminaries

A ***ranking*** of a nonempty finite set X is a bijection $r : X \to \{1, \dots, |X|\}$. We interpret $r(x) < r(x')$ as x being preferred to x'. Thus, rank 1 is best, rank 2 is second-best, etc.

A ***marriage market*** is a tuple (M, W, R_M, R_W) where M and W are disjoint sets of finite size $n > 0$ of ***men*** and ***women***, called the two ***sides*** of the market, $R_M = (r_m)_{m \in M}$ is an n-tuple of rankings of W by the men, and $R_W = (r_w)_{w \in W}$ is an n-tuple of rankings of M by the women. We refer also to R_M and R_W as the *sets* of rankings in each n-tuple correspondingly. No confusion will result.

A ***matching*** is a set of pairs $\mu = \{(m, w)\}$ which is the graph of a bijection of M and W. For each man m we denote by $\mu(m)$ the unique woman w such that $(m, w) \in \mu$. For each woman w, $\mu(w)$ is similarly defined.

[1] Lee's result can be reconciled with Pittel's by noting that, as $n \to \infty$, if n ranks are scaled to lie in a fixed interval, then $\frac{n}{\ln n}$ consecutive ranks occupy only a vanishing subinterval. We note that Lee's model allows for correlation in preferences, but this is not what drives his result (which holds regardless of the correlation).

A pair (m, w) ***blocks*** the matching μ if $r_m(w) < r_m(\mu(m))$ and $r_w(m) < r_w(\mu(w))$. The matching μ is ***stable*** if no pair blocks it. The core, C, of the marriage problem is the set of all its stable matchings. There exists a man-optimal stable matching, μ_M, that satisfies for each m and w, $r_m(\mu_M(m)) = \min_{\mu \in C} r_m(\mu(m))$, and $r_w(\mu_M(w)) = \max_{\mu \in C} r_w(\mu(w))$. Similarly, there exists a woman-optimal stable matching μ_W with the corresponding properties.

We say that one of the sides of a marriage market is ***universally ranked*** if it is ranked in the same way by all the individuals of the other side. If, say, the men are universally ranked as $m_1, \dots, m_n$, then it is easy to check that in any stable matching m_1 must be matched to his top choice, m_2 must be matched to his highest choice among the remaining $n-1$ women, and so on. This leads to the following consequences:

(i) If one of the sides, say M, is universally ranked, then there exists a unique stable matching.
(ii) In this matching each man ranks his spouse no worse than she ranks him.
(iii) Consequently, when both sides are universally ranked, then individuals who are matched in the unique stable matching, have the same rank.

3. The size of the ranking sets and the rank gaps

Given a set R of rankings of a set X, the ***displacement*** of $x \in X$ is $\delta(x) = \max_{r \in R} r(x) - \min_{r \in R} r(x)$. We use the maximal displacement, $\Delta^{\max}(R) = \max_{x \in X} \delta(x)$ and the average displacement, $\Delta^{\text{av}}(R) = (1/n) \sum_{x \in X} \delta(x)$, where $n = |X|$, as measures of the size of R.

The ***rank gap*** of a pair $(m, w) \in M \times W$ is $\gamma(m, w) = |r_m(w) - r_w(m)|$. The disparity of the mutual rankings of spouses in a given matching μ is measured by the maximal rank gap in μ, $\Gamma^{\max}(\mu) = \max_{(m,w) \in \mu} \gamma(m, w)$ and the average rank gap in μ, $\Gamma^{\text{av}}(\mu) = (1/n) \sum_{(m,w) \in \mu} \gamma(m, w)$.

Our first theorem shows that property (ii) above is robust: if the men are not quite universally ranked, but every man is displaced across R_W by at most k ranks, then in every stable matching, a man may rank his spouse worse than she ranks him by no more than $2k$ ranks. The case $k = 0$ of this statement reduces to property (ii).

Theorem 1. *For each stable matching μ and $(m, w) \in \mu$,*

$$r_m(w) - r_w(m) \le 2\Delta^{\max}(R_W). \tag{1}$$

Proof. Let μ be a stable matching and $(m, w) \in \mu$. Man m prefers $r_m(w) - 1$ women to w. By the stability of μ each one of these $r_m(w) - 1$ women is matched to a man she prefers to m. Thus, we have $r_m(w) - 1$ distinct men m', each of them preferred to m by at least one woman. For every such m', the bound on the displacements of m and m' implies that

$$r_w(m') < r_w(m) + 2\Delta^{\max}(R_W). \tag{2}$$

In words, man m' must be among the $r_w(m) + 2\Delta^{\max}(R_W) - 1$ top choices of woman w. As we have $r_m(w) - 1$ men m', it follows that

$$r_m(w) - 1 \le r_w(m) + 2\Delta^{\max}(R_W) - 1. \tag{3}$$

Inequality (1) follows.[2] □

As a corollary of Theorem 1, we find that property (iii) above is robust: if the rankings are not quite universal, but every individual is displaced across the rankings by the members of the other side by at most k ranks, then in every stable matching, the maximal rank gap is no more than $2k$. The case $k = 0$ of this statement reduces to property (iii).

Corollary 1. *For any stable matching μ,*

$$\Gamma^{\max}(\mu) \le 2 \max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\}. \tag{4}$$

Proof. By (1) and the analogous bound $r_w(m) - r_m(w) \le 2\Delta^{\max}(R_M)$ we conclude that for each stable matching μ and pair $(m, w) \in \mu$, $|r_w(m) - r_m(w)| \le 2 \max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\}$, from which (4) follows. □

When the maximal displacement is much larger than the average one, the upper bound on the maximal rank gap obtained above may not be useful. Hence we proceed to establish an upper bound in terms of average displacements. Namely, in any stable matching, the average rank gap is at most twice the sum of the average displacement of men and the average displacement of women.

[2] For $\Delta^{\max}(R_W) \ge 1$ a tighter bound holds, namely $r_m(w) - r_w(m) \le 2\Delta^{\max}(R_W) - 1$. Indeed, in this case one of the $r_w(m) + 2\Delta^{\max}(R_W) - 1$ top choices of woman w is m himself. Thus, taking into account that all men m' in question are distinct from m, we gain 1 in inequality (3).

Theorem 2. *For any stable matching* μ*,*

$$\Gamma^{\mathrm{av}}(\mu) \le 2\big(\Delta^{\mathrm{av}}(R_W) + \Delta^{\mathrm{av}}(R_M)\big). \tag{5}$$

In the proof of Theorem 2, we break the sum of rank gaps in a stable matching into two parts: the sum over those matched pairs in which the woman ranks her husband better than he ranks her, and the sum over those pairs where the opposite holds. In order to upper bound each of these two sums separately, we use the following proposition.

Proposition 1. *For every stable matching* μ *and any subset* M_0 *of* M*,*

$$\sum_{m \in M_0} \big[r_m\big(\mu(m)\big) - r_{\mu(m)}(m)\big] \le \sum_{m \in M_0} \delta(m) + \sum_{m' \in M} \delta\big(m'\big) \le 2n\Delta^{\mathrm{av}}(R_W). \tag{6}$$

Proof. Let μ be a stable matching and $M_0 \subseteq M$. Consider a man $m \in M_0$. As in the proof of Theorem 1, by the stability of μ, we can find $r_m(\mu(m)) - 1$ men m', each of them satisfying

$$r_{\mu(m')}\big(m'\big) < r_{\mu(m')}(m). \tag{7}$$

Given any fixed woman and any rank i, she can rank at most $i-1$ of those men m' in ranks $1, \ldots, i-1$. Taking $i = \max_{w \in W} r_w(m)$, we conclude that at least $r_m(\mu(m)) - \max_{w \in W} r_w(m)$ of these men m' are ranked by her in rank $\max_{w \in W} r_w(m)$ or worse, thus satisfying

$$\max_{w \in W} r_w\big(m'\big) \ge \max_{w \in W} r_w(m). \tag{8}$$

Denote by P_m the set of men m' satisfying (7) and (8). As shown, we have $|P_m| \ge r_m(\mu(m)) - \max_{w \in W} r_w(m)$. Doing this for each $m \in M_0$ separately, we get a system of sets P_m, $m \in M_0$, with union $P = \bigcup_{m \in M_0} P_m$. For each $m' \in P$, let $Q_{m'} = \{m \in M_0 \mid m' \in P_m\}$. Such a man m' satisfies (7) with respect to every $m \in Q_{m'}$, and therefore

$$r_{\mu(m')}\big(m'\big) \le \max_{m \in Q_{m'}} r_{\mu(m')}(m) - |Q_{m'}| \le \max_{m \in Q_{m'}} \max_{w \in W} r_w(m) - |Q_{m'}|. \tag{9}$$

On the other hand, since m' satisfies (8) with respect to every $m \in Q_{m'}$, we get

$$\max_{w \in W} r_w\big(m'\big) \ge \max_{m \in Q_{m'}} \max_{w \in W} r_w(m). \tag{10}$$

Combining (9) and (10), we obtain that $\delta(m') \ge |Q_{m'}|$. This yields

$$\begin{aligned}\sum_{m \in M_0} \Big[r_m\big(\mu(m)\big) - \max_{w \in W} r_w(m)\Big] &\le \sum_{m \in M_0} |P_m| = \sum_{m' \in P} |Q_{m'}| \\ &\le \sum_{m' \in P} \delta\big(m'\big) \le \sum_{m' \in M} \delta\big(m'\big).\end{aligned} \tag{11}$$

We also have

$$\sum_{m \in M_0} \Big[\max_{w \in W} r_w(m) - r_{\mu(m)}(m)\Big] \le \sum_{m \in M_0} \delta(m), \tag{12}$$

and upon adding (11) and (12) we get (6). □

Proof of Theorem 2. Given a stable matching μ, let M_0 be the set of men m for whom $r_m(\mu(m)) > r_{\mu(m)}(m)$, and let W_0 be the set of women w for whom $r_w(\mu(w)) > r_{\mu(w)}(w)$. Adding (6) and the analogous bound for the subset W_0 of W, and dividing by n, we obtain (5). □

Theorem 2 shows that the average rank gap is guaranteed to be small even if there are a few individuals about whom there is significant disagreement in the rankings, as long as the rankings roughly agree about most individuals.

A different kind of concern about the applicability of our bounds involves the possible presence of a few individuals who hold significantly different rankings. Indeed, suppose that most men rank the women in roughly the same way, but there is a small set of men M^* whose rankings of the women are out-of-line with the rest. Similarly, most women rank the men in roughly the same way, but there is a small set of women W^* with very different rankings of the men. Then Theorem 2 is not directly useful, because the right-hand-side is made large by the presence of M^* and W^*. Nevertheless, the result is robust with respect to this possibility. All we need to do is to consider the average displacements across the rankings by "regular" individuals only, namely, $\Delta^{\mathrm{av}}(R_{W \setminus W^*})$ and $\Delta^{\mathrm{av}}(R_{M \setminus M^*})$. We get the bound (5) up to a small error term:

$$\Gamma^{\mathrm{av}}(\mu) \le 2\big(\Delta^{\mathrm{av}}(R_{W \setminus W^*}) + \Delta^{\mathrm{av}}(R_{M \setminus M^*})\big) + |W^*| + |M^*| + \max\big(|W^*|, |M^*|\big).$$

The proof is a straightforward adaptation of the proofs of Proposition 1 and Theorem 2, and is omitted.

4. The size of the core

We now provide bounds on the size of the core in terms of the size of the ranking sets and the rank gap. For this we define two metrics on matchings. The ***woman-metric*** on matchings, d_W, is defined for each pair of matchings μ_1 and μ_2 by

$$d_W(\mu_1,\mu_2)=(1/n)\sum_{w\in W}\big|r_w\big(\mu_1(w)\big)-r_w\big(\mu_2(w)\big)\big|.$$

The ***man-metric*** d_M is similarly defined. The diameters of the core with respect to the metrics d_W and d_M are denoted by $D_W(C)$ and $D_M(C)$ correspondingly. For stable matchings μ_1 and μ_2, $|r_w(\mu_1(w))-r_w(\mu_2(w))|\le r_w(\mu_M(w))-r_w(\mu_W(w))$ for each w. Thus, $D_W(C)=(1/n)\sum_{w\in W}[r_w(\mu_M(w))-r_w(\mu_W(w))]$, and a similar expression holds for $D_M(C)$.

The following theorem shows that property (i) in Section 2 is robust: if the men are not quite universally ranked, but their average displacement across R_W is known to be small, then the diameter of the core with respect to the woman-metric is not larger than that. The case $\Delta^{\mathrm{av}}(R_W)=0$ of this statement reduces to property (i).

Theorem 3.

$$D_W(C)\le\Delta^{\mathrm{av}}(R_W).$$

Proof.

$$\begin{aligned}D_W(C)&=(1/n)\sum_{w\in W}\big[r_w\big(\mu_M(w)\big)-r_w\big(\mu_W(w)\big)\big]\\&=(1/n)\sum_{m\in M}\big[r_{\mu_M(m)}(m)-r_{\mu_W(m)}(m)\big]\\&\le(1/n)\sum_{m\in M}\delta(m)\\&=\Delta^{\mathrm{av}}(R_W).\qquad\square\end{aligned}$$

Similar to the discussion at the end of the previous section, Theorem 3 can also be adapted to allow for the presence of a few individuals holding out-of-line rankings. A straightforward adaptation of the proof above yields:

$$D_W(C)\le\Delta^{\mathrm{av}}(R_{W\setminus W^*})+2\big|W^*\big|.$$

In the next theorem, the size of the core is bounded in terms of the average gap of the woman and man optimal matchings.

Theorem 4.

$$D_M(C)+D_W(C)\le\Gamma^{\mathrm{av}}(\mu_M)+\Gamma^{\mathrm{av}}(\mu_W).$$

Proof. Define $S_{MM}=\sum_{m\in M}r_m(\mu_M(m))$ and $S_{MW}=\sum_{m\in M}r_m(\mu_W(m))$, and define S_{WW} and S_{WM} similarly. Then $D_M(C)=(1/n)[S_{MW}-S_{MM}]$ and $D_W(C)=(1/n)[S_{WM}-S_{WW}]$. Next, observe that

$$\begin{aligned}|S_{WM}-S_{MM}|&=\Big|\sum_{w\in W}r_w\big(\mu_M(w)\big)-\sum_{m\in M}r_m\big(\mu_M(m)\big)\Big|\\&=\Big|\sum_{w\in W}r_w\big(\mu_M(w)\big)-\sum_{w\in W}r_{\mu_M(w)}(w)\Big|\\&\le\sum_{w\in W}\big|r_w\big(\mu_M(w)\big)-r_{\mu_M(w)}(w)\big|\\&=n\Gamma^{\mathrm{av}}(\mu_M),\end{aligned}$$

and similarly, $|S_{MW}-S_{WW}|\le n\Gamma^{\mathrm{av}}(\mu_W)$. Thus,

$$\begin{aligned}D_M(C)+D_W(C)&=(1/n)[S_{MW}-S_{MM}+S_{WM}-S_{WW}]\\&\le(1/n)\big[|S_{WM}-S_{MM}|+|S_{MW}-S_{WW}|\big]\\&\le\Gamma^{\mathrm{av}}(\mu_M)+\Gamma^{\mathrm{av}}(\mu_W).\qquad\square\end{aligned}$$

The following is an immediate corollary of this theorem.

Corollary 2. *If the rank gaps in the man-optimal and the woman-optimal matchings vanish, then there exists a unique stable matching.*

5. Examples and counterexamples

We present here a few constructions of marriage markets, showing that some of the bounds proved above are tight, and indicating that certain variants of these bounds do not hold in general.

Our first example shows that the upper bounds in Theorem 1 and Corollary 1 are tight (in the slightly improved form given in footnote 2).

Example 1. Let $k \geq 1$. Consider a market with $2k$ individuals on each side, numbered as $M = \{m_1, \dots, m_{2k}\}$ and $W = \{w_1, \dots, w_{2k}\}$. Let the women be universally ranked from top to bottom as $w_1, \dots, w_{2k}$. Let the rankings of the men by the women be as follows:

$$w_i: m_1, m_2, \dots, m_k, m_{2k}, m_{k+1}, \dots, m_{2k-1} \quad (i = 1, \dots, k)$$

$$w_j: m_1, m_{k+1}, \dots, m_{2k-1}, m_{2k}, m_2, \dots, m_k \quad (j = k+1, \dots, 2k-1)$$

$$w_{2k}: m_{2k}, m_1, \dots, m_{k-1}, m_k, m_{k+1}, \dots, m_{2k-1}$$

The unique stable matching is obtained when each of the women $w_1, \dots, w_{2k}$ in turn gets her top choice among the still available men. This yields the matching $\{(m_i, w_i)\}_{i=1,\dots,2k}$, with $r_{m_{2k}}(w_{2k}) - r_{w_{2k}}(m_{2k}) = 2k - 1$. On the other hand, it is easy to check that $\Delta^{\max}(R_W) = k$. This shows that the upper bound $r_m(w) - r_w(m) \leq 2\Delta^{\max}(R_W) - 1$ (for $\Delta^{\max}(R_W) \geq 1$) is tight. As $\Delta^{\max}(R_M) = 0$, this example also shows that one cannot replace the upper bound $2\max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\}$ on $\Gamma^{\max}(\mu)$ by $\Delta^{\max}(R_W) + \Delta^{\max}(R_M)$.

For the bound on the average rank gap in terms of the average displacements, we do not have a construction meeting the upper bound. In fact, we conjecture that the factor of 2 in the upper bounds of Proposition 1 and Theorem 2 can be lowered to 1. The following example shows that it cannot be replaced by any constant factor smaller than 1.

Example 2. Consider a market with $M = \{m_1, \dots, m_n\}$ and $W = \{w_1, \dots, w_n\}$. Let the women be universally ranked as $w_1, \dots, w_n$. Let the ranking of the men by woman w_i, $i = 1, \dots, n$, be obtained from the ranking $m_1, \dots, m_n$ by promoting m_i to the top of the list, while leaving the other men in the same order. The unique stable matching is $\{(m_i, w_i)\}_{i=1,\dots,n}$. Here the rank gaps are $0, 1, \dots, n-1$ respectively, while the displacements of the men are $1, 2, \dots, n-1, n-1$ respectively. Thus $\Gamma^{\text{av}}(\mu) = (n-1)/2$, $\Delta^{\text{av}}(R_W) = (n+2)(n-1)/(2n)$, and the ratio between them approaches 1 as n goes to infinity.

According to statement (i) above, if *either one* of the sides is universally ranked, then the core is a singleton. Thus, one may expect to be able to assert that the diameter of the core in the woman-metric, $D_W(C)$, is small, not only when $\Delta^{\text{av}}(R_W)$ is small (as shown in Theorem 3), but also when $\Delta^{\text{av}}(R_M)$ is small. The following example refutes this intuition, and illustrates some additional points that we discuss below.

Example 3. Let $k \geq 2$, and let n be a multiple of k, say $n = k\ell$. Consider a market where the men are partitioned into ℓ blocks of size k each: $M^i = \{m^i_1, \dots, m^i_k\}$, $i = 1, \dots, \ell$. Similarly, the women are partitioned into $W^i = \{w^i_1, \dots, w^i_k\}$, $i = 1, \dots, \ell$. Let every man rank the blocks of women as $W^1, \dots, W^\ell$; within the blocks, the women are ranked as $w^i_1, \dots, w^i_k$, except that for each i, the men in M^i rank the women in the corresponding block W^i in a cyclic fashion:

$$m^i_j: w^i_j, \dots, w^i_k, w^i_1, \dots, w^i_{j-1}$$

(with subscripts taken modulo k). Every woman in W^i, $i = 1, \dots, \ell$, ranks $\bigcup_{p=1}^{i} M^p$ above the rest of the men; within this union of blocks, woman w^i_j ranks m^i_{j+1} first and m^i_j last (that is, in rank ik); besides that, the rankings are immaterial.

One may check, by induction on i, that in every stable matching the men in M^i are matched to the women in W^i, $i = 1, \dots, \ell$. Within each pair of blocks M^i, W^i, the man-optimal stable matching μ_M consists of the pairs $\{(m^i_j, w^i_j)\}_{j=1,\dots,k}$, whereas the woman-optimal one μ_W consists of the pairs $\{(m^i_{j+1}, w^i_j)\}_{j=1,\dots,k}$. Thus,

$$D_W(C) = \frac{1}{k\ell} \sum_{i=1}^{\ell} k(ik - 1) = \frac{k(\ell+1)}{2} - 1.$$

Note that $\Delta^{\text{av}}(R_M) = \Delta^{\max}(R_M) = k - 1$. By keeping k fixed and letting ℓ grow, we see that $D_W(C)$ cannot be bounded by any function of $\Delta^{\text{av}}(R_M)$ or even $\Delta^{\max}(R_M)$. As remarked above, if $\Delta^{\max}(R_M)$ vanishes then so does $D_W(C)$, but our construction shows that any positive value of $\Delta^{\max}(R_M)$ is consistent with arbitrarily large values of $D_W(C)$.

We observe also that in our example $D_M(C) = k-1$, which shows (upon interchanging the roles of men and women) that Theorem 3 is tight. It may also be checked that our example gives equality in Theorem 4, thus showing its tightness, as well.

Example 3 serves to illustrate yet another point. A different way to measure the size of a set R of rankings of a set X would be to define a metric on R by $d(r,r') = (1/n)\sum_{x\in X}|r(x)-r'(x)|$, where $n=|X|$, and consider $D(R) = \max_{r,r'\in R} d(r,r')$, the diameter of R under this metric. Note that in general $D(R) \le \Delta^{\mathrm{av}}(R)$. To calculate $D(R_M)$ in our example, note that if $m\in M^i$ and $m'\in M^{i'}$ then r_m and $r_{m'}$ may differ only regarding women in $W^i\cup W^{i'}$. Within W^i one ranking is a cyclic shift of the other, so in the worst case we have $\sum_{w\in W^i}|r_m(w)-r_{m'}(w)| = \lfloor k^2/2\rfloor$, and similarly for $W^{i'}$. This gives

$$D(R_M) = \frac{1}{k\ell}\cdot 2\left\lfloor\frac{k^2}{2}\right\rfloor \le \frac{k}{\ell}.$$

By keeping the ratio k/ℓ fixed while both of them grow, we see that $D(R_M)$ can be arbitrarily small while $D_M(C)$ is arbitrarily large. We conclude that this alternative measure of the size of a set of rankings cannot replace Δ^{av} in providing an upper bound on the size of the core (or, for that matter, on the average rank gap).

Our final question is whether there exist upper bounds, similar to Theorems 3 and 4, not only on the rank difference between mates in μ_M and μ_W for an average individual, but for *every* individual. The following example gives a negative answer.

Example 4. Consider a market with $M=\{m_1,\dots,m_n\}$ and $W=\{w_1,\dots,w_n\}$. The women are basically ranked as $w_2,\dots,w_n,w_1$, but man m_i, $i=2,\dots,n-1$, swaps w_i and w_{i+1} in his ranking, and the other two men make specific adjustments as indicated:

$$m_1:\ w_2,w_1,w_3,\dots,w_n$$

$$m_2:\ w_3,w_2,w_4,\dots,w_n,w_1$$

$$m_i:\ w_2,\dots,w_{i-1},w_{i+1},w_i,w_{i+2},\dots,w_n,w_1 \quad (i=3,\dots,n-2)$$

$$m_{n-1}:\ w_2,\dots,w_{n-2},w_n,w_{n-1},w_1$$

$$m_n:\ w_2,\dots,w_{n-1},w_1,w_n$$

The men are basically ranked as $m_1,\dots,m_n$, but woman w_i, $i=2,\dots,n$, swaps m_{i-1} and m_i in her ranking, yielding the rankings:

$$w_1:\ m_1,m_2,\dots,m_n$$

$$w_2:\ m_2,m_1,m_3,\dots,m_n$$

$$w_i:\ m_1,\dots,m_{i-2},m_i,m_{i-1},m_{i+1},\dots,m_n \quad (i=3,\dots,n-1)$$

$$w_n:\ m_1,\dots,m_{n-2},m_n,m_{n-1}$$

We claim that $\mu = \{(m_i,w_{i+1})\}_{i=1,\dots,n-1}\cup\{(m_n,w_1)\}$ is the man-optimal stable matching. To check stability, note that if m_i prefers w_j to his mate then $2\le j\le i-1$, but such a woman w_j prefers her mate to m_i. To verify that μ is man-optimal, use the fact that μ_M must satisfy $r_w(\mu_M(w)) \ge r_w(\mu(w))$ for every $w\in W$. Considering in turn the women $w_1,w_n,w_{n-1},\dots$, this forces $\mu_M=\mu$.

Next, we claim that $\mu' = \{(m_i,w_i)\}_{i=1,\dots,n}$ is the woman-optimal stable matching. To check stability, note that if w_i prefers m_j to her mate then $j\le i-2$, but such a man m_j prefers his mate to w_i. To verify that μ' is woman-optimal, use the property $r_w(\mu_W(w)) \le r_w(\mu'(w))$ successively for the women $w_1,w_2,w_3,\dots$, deducing that $\mu_W=\mu'$.

Now, woman w_1 is matched to her top-ranked man m_1 in μ_W and to her bottom-ranked man m_n in μ_M. This is in spite of the fact that $\delta(m)\le 2$ for every $m\in M$, which shows that the upper bound of Theorem 3 does not hold when both sides of the inequality are replaced by their max versions. A similar conclusion applies to the bound of Theorem 4, since $\Gamma^{\max}(\mu_M)=2$ and $\Gamma^{\max}(\mu_W)=1$.

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