The Domain of RSD Characterization by Efficiency, Symmetry, and Strategy-Proofness

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Abstract

Given a set of n individuals with strict preferences over m indivisible objects, the Random Serial Dictatorship (RSD) mechanism is a method for allocating at most one object to each individual in a way that is efficient, fair, and incentive-compatible. A random order of individuals is first drawn, and each individual, following this order, selects their most preferred available object. The procedure continues until either all objects have been assigned or all individuals have received an object.

RSD is widely recognized for its application in fair allocation problems involving indivisible goods, such as school placements and housing assignments. Despite its extensive use, a comprehensive axiomatic characterization has remained incomplete. For the balanced case $n = m = 3$, Bogomolnaia and Moulin have shown that RSD is uniquely characterized by Ex-Post Efficiency, Equal Treatment of Equals, and Strategy-Proofness. The possibility of extending this characterization to larger markets had been a long-standing open question, which Basteck and Ehlers recently answered in the negative for all markets with $n, m \geq 5$.

This work completes the picture by identifying exactly for which pairs (n, m) these three axioms uniquely characterize the RSD mechanism and for which pairs they admit multiple mechanisms. In the latter cases, we construct explicit alternatives satisfying the axioms and examine whether augmenting the set of axioms could rule out these alternatives.

1 Introduction

The problem of assigning indivisible objects to agents in a fair and efficient way is central in mechanism design. Typical applications include school choice, assignment of students to dorm rooms, and allocation of public housing. In such environments, monetary transfers are often unavailable or undesirable, and the only information agents report is a ranking of the available

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objects. A mechanism must then map these preference profiles into allocations that satisfy normative criteria such as efficiency, fairness, and incentive compatibility. Since deterministic rules often fail to meet fairness requirements even when considered on their own, and in particular cannot satisfy efficiency, fairness, and incentive compatibility simultaneously, the literature has focused on randomized mechanisms, which assign to each agent a lottery over the objects, with an outside option of receiving nothing. Throughout, we assume that every agent ranks this outside option below every object.

A particularly prominent mechanism in this context is Random Serial Dictatorship (RSD), also known in the literature as Random Priority. Given a profile of strict preferences and a set of indivisible objects, RSD samples an ordering of the agents uniformly at random and lets them choose their most preferred available object in that order; if there are more agents than objects, the last agents receive nothing, whereas if there are more objects than agents, some objects remain unassigned. In the random assignment literature, RSD is an extensively studied mechanism. For example, Abdulkadiroğlu and Sönmez [1] and, independently, Knuth [13] have shown that RSD coincides with the core from random endowments, and Bade [3] has shown that symmetrizing any deterministic efficient mechanism satisfying strategy-proofness and non-bossiness yields RSD.

RSD satisfies ex-post efficiency, equal treatment of equals, and strategy-proofness (these three properties will be referred to as *the axioms*). Ex-post efficiency requires that, for every preference profile, the mechanism only randomizes over deterministic assignments that are Pareto efficient; equal treatment of equals requires that agents with identical preferences receive identical lotteries; and strategy-proofness requires that truthful reporting is a weakly dominant strategy for every agent. These three axioms largely explain its popularity in practice. At the same time, it is natural to consider stronger notions of efficiency and fairness, and this led to the study of alternative mechanisms alongside RSD. Bogomolnaia and Moulin [8] proposed the Probabilistic Serial (PS) mechanism as an alternative to RSD.[1] Beyond verifying that a given mechanism satisfies desirable axioms, an important question in mechanism design is whether a given set of axioms characterizes that mechanism. For example, Bogomolnaia and Heo [7], Hashimoto et al. [11], and Bogomolnaia [6] provide axiomatic characterizations of the PS mechanism.

In this work we focus on the question of when the axioms uniquely characterize RSD. For the balanced case with three agents and three objects, Bogomolnaia and Moulin [8] showed that the axioms uniquely characterize RSD. This raised a natural question: do the axioms single out RSD for all numbers of agents and objects? This question remained open for a long time and attracted sustained attention. Its difficulty also motivated work on related problems, including axiomatic characterizations of RSD in related frameworks. Basteck [4] has shown that, when randomized mechanisms are viewed as assigning lotteries over deterministic assignments to each profile, RSD can be characterized by ex-post efficiency, equal treatment of equals, and probabilistic (Maskin) monotonicity; and Pycia and Troyan [16] have represented randomized mechanisms as an extensive-form game and have obtained such a characterization by strengthening strategy-proofness to obvious

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[1] PS satisfies ordinal efficiency and envy-freeness, which are stronger than ex-post efficiency and equal treatment of equals, respectively, whereas RSD does not; on the other hand, PS does not satisfy strategy-proofness, but only a weaker version of it.

strategy-proofness.[2] The difficulty of this question also spurred work on the limits of the axiom system; in particular, each of the following one-axiom strengthenings leads to an impossibility result, in the sense that no mechanism satisfies the resulting strengthened axiom system: Bogomolnaia and Moulin [8] strengthen ex-post efficiency to ordinal efficiency; in a different setting with cardinal utilities, Zhou [18] strengthens ex-post efficiency to ex-ante efficiency; Nesterov [15] strengthens equal treatment of equals to envy-freeness; and Bade [2] strengthens strategy-proofness to group strategy-proofness.

At the center of attention was the question whether the axioms uniquely characterize RSD in the balanced case where the number of objects equals the number of agents. In that line of work, the case of four agents and four objects became a touchstone: it was the first unsolved balanced market size, and it resisted purely analytic proofs, being resolved only via a computer-assisted analysis (Sandomirskiy [17] and unpublished computerized verification by others). Finally, the long-standing question was answered in the negative by Basteck and Ehlers [5], who constructed mechanisms different from RSD that still satisfy the axioms and showed that such mechanisms exist for all markets with at least five agents and at least five objects. Their construction is inspired by a result of Erdil [10], who has shown that if agents are allowed to rank receiving nothing above some objects, then mechanisms other than RSD exist that satisfy the axioms.

The present work completes this picture. We fix a number of agents n and a number of indivisible objects m with $n, m \geq 2$, and consider the randomized mechanisms that satisfy the axioms. Our main result is a complete classification of the pairs (n, m) according to whether the axioms uniquely characterize RSD or whether they admit additional distinct mechanisms. On the positive side, we show that the axioms uniquely determine the assignment probabilities for every preference profile whenever there are at most three agents and an arbitrary number of objects, and in the balanced case with four agents and four objects. On the negative side, we show that outside this domain the axioms are too weak: for each remaining pair (n, m) we show how to construct mechanisms that satisfy the axioms, yet differ from RSD. In retrospect, the full classification obtained in the present work helps explain why the balanced case $n = m = 4$ proved difficult: when $n = 4$, uniqueness holds only in the balanced case $m = 4$, a phenomenon that does not arise for other values of n. Table 1 summarizes the uniqueness and non-uniqueness results.

For markets with at least three agents in the domain where the axioms uniquely characterize RSD, our proofs rely on a local analysis of the axioms, using adjacent swaps in agents' rankings to propagate constraints across profiles and show that the assignment matrix is uniquely determined for every preference profile. Notably, these uniqueness arguments only require weakened versions of ex-post efficiency and strategy-proofness. First, the weakened version of ex-post efficiency required in our uniqueness arguments is support efficiency (in the terminology of Brandt et al. [9]), together with the full assignment property (Definition 2.9). Support efficiency requires that an agent can have a positive probability of receiving a given object only if he receives it under some Pareto efficient deterministic assignment, and the full assignment property requires that whenever an agent (respectively, an object) is matched in every serial dictatorship outcome, the mechanism matches that agent (respectively, allocates that object) with probability 1. Second, in place of full

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[2] In each of these two characterizations, the key axiom (probabilistic monotonicity, obvious strategy-proofness) is one that cannot be expressed in the minimalistic setting adopted here, where a mechanism maps preference profiles to probability matrices.

n m	2	3	4	≥ 5
2	✓ (Section 3)		✗ (Section 3)	
3	✓ (Remark 10)	✓ [8]	✗ (Section 5.1)	
4		✓ (Section 4.1)	✓ (Section 4.2 & Appendix B)	
≥ 5			✗ (Section 5.2)	✗ [5]

Table 1: Do the axioms uniquely determine RSD with n agents and m objects? (✓ yes, ✗ no)

strategy-proofness, it suffices to assume upper invariance and lower invariance (in the terminology of Mennle and Seuken [14]), two conditions which, together with swap monotonicity, are equivalent to strategy-proofness. Upper and lower invariance mean that when an agent swaps two adjacent objects in his ranking, his probabilities of receiving any other object, as well as the outside option, remain unchanged.

We also provide a complete description of the case $m = 2$. In this case, we show that every mechanism satisfying the axioms can be represented by a single function that assigns to each subset of agents the probability with which its members receive their preferred object, when they rank the two objects the same way, and the other agents rank them the opposite way. This yields a simple parameterization of the entire class of mechanisms satisfying the axioms and allows us to determine exactly when RSD is the unique mechanism and when there is a continuum of alternatives.

In the non-uniqueness domain we follow a different strategy, conceptually close in spirit to the construction of Basteck and Ehlers [5]. We first identify RSD as the symmetrization of a specific mechanism that satisfies ex-post efficiency and strategy-proofness. We then perform carefully designed adjustments to this underlying mechanism on a restricted family of profiles, chosen so that these two axioms remain satisfied. Symmetrizing the adjusted mechanism yields a mechanism that satisfies all three axioms yet is distinct from RSD.

Building on the work of Basteck and Ehlers [5], whose construction yields a mechanism that satisfies the axioms and, in addition, anonymity (a strengthening of equal treatment of equals) and neutrality, we then investigate whether further strengthening the axiom system can restore uniqueness. In particular, we consider adding the bounded invariance axiom and answer their open question on whether this addition can restore uniqueness of RSD in the negative. We then go further by additionally imposing non-bossiness and cross monotonicity, and show that even with these extra axioms, the resulting system still does not uniquely characterize RSD in sufficiently large markets.

The underlying reason these alternative mechanisms can exist is structural: in many markets, one can identify a family of profiles where RSD satisfies adjacent-swap constraints (such as those required by strategy-proofness and bounded invariance) with some slack. This slack arises either because a required inequality holds strictly, allowing for slight probability adjustments without violating the condition, or because certain house-agent probabilities are entirely unconstrained during a specific swap. By exploiting this local freedom, one can introduce carefully designed perturbations to RSD that remain invisible to axioms based on adjacent-swap comparisons, ultimately yielding distinct mechanisms.

The work is organized as follows. Section 2 introduces the formal model, recalls the RSD mechanism together with the axioms of ex-post efficiency, equal treatment of equals, and strategy-proofness, and establishes several preliminary lemmas, including characterizations of ex-post efficiency and strategy-proofness and further properties implied by the axioms that are used later in the proofs. In Section 3 we analyze the case of two objects and obtain a complete parameterization of all mechanisms satisfying the axioms in this setting. Section 4 contains the positive results identifying all market sizes for which the axioms uniquely determine RSD. Section 5 provides the non-uniqueness results by constructing explicit alternative mechanisms for all remaining values of n and m, and shows that adding the bounded invariance axiom does not restore uniqueness of RSD. Finally, the appendix consists of two parts. Appendix A introduces the non-bossiness and cross monotonicity axioms, and shows that even adding these axioms on top of the previous ones does not restore uniqueness of RSD in sufficiently large markets. Appendix B presents the exhaustive case analysis for the market with four agents and four objects.

2 Preliminaries

2.1 The model

We begin with some preliminary definitions. Let N denote the set of individuals, referred to as *agents*, and H the set of indivisible objects, referred to as *houses*. We define their sizes as $|N| \coloneqq n$ and $|H| \coloneqq m$, and identify the set of agents with $[n] = \{1, \ldots, n\}$.

We consider an environment in which each house can be assigned to at most one agent, and each agent can receive at most one house. Since n and m may differ, we define an *assignment* as a matching in the complete bipartite graph with vertex set $N \cup H$, and let $\mathcal{S}$ denote the set of all such assignments. An assignment $s \in \mathcal{S}$ *assigns* agent $i \in N$ to house $h \in H$ if $\{i, h\} \in s$, and we write $s(i) \coloneqq h$. If i is not assigned any house under s, we say that i is *unassigned* and write $s(i) \coloneqq \varnothing$, where $\varnothing \notin H$ denotes the *null object*. We define the set of all possible objects as $O \coloneqq H \cup \{\varnothing\}$. From this point on, we use *object* to refer to an element of O (including the null object), and *house* to refer specifically to an element of H.

We consider settings where each agent's preference depends only on the object they are assigned, rather than on the full assignment. A *preference order* is a total order over the set O, in which every house $h \in H$ is strictly preferred to the null object $\varnothing$. Let $\mathcal{R}$ denote the set of all preference orders. For a preference order $R \in \mathcal{R}$ and two objects $o, o' \in O$ the notation $o' R o$ denotes that o' is weakly preferred to o under R, and $o' R^{+} o$ denotes that o' is strictly preferred to o under

R. The *upper contour set* of o with respect to R is defined as

$$C_R(o) \coloneqq \{o' \in O \mid o' R o\}.$$

In other words, $C_R(o)$ is the set of objects that are at least as good as o according to the preference order R.

Since we will consider randomized mechanisms, agents are assigned probability distributions over objects rather than the objects themselves. For a preference order $R \in \mathcal{R}$, we extend the preference relation to a partial order over the set $\Delta(O)$ of probability distributions over O, using *first-order stochastic dominance.* Specifically, for $p, p' \in \Delta(O)$, we say that an agent with preference order R *weakly prefers* p to p', denoted $p \succeq_R p'$, if for every object $o \in O$,

$$\sum_{o' \in C_R(o)} p(o') \geq \sum_{o' \in C_R(o)} p'(o').$$

We say that the agent *strictly prefers* p to p', denoted $p \succ_R p'$, if $p \succeq_R p'$ and the inequality is strict for at least one $o \in O$.

A *preference profile* is an element $\mathbf{P} = (P_1, \ldots, P_n) \in \mathcal{R}^N$ that assigns to each agent $i \in N$ a preference order $P_i \in \mathcal{R}$. For $o, o' \in O$, if $o P_i o'$, we say that agent i *weakly prefers* o to o' under the profile $\mathbf{P}$.

A *deterministic mechanism* is a function $\mathcal{R}^N \to \mathcal{S}$, assigning to each preference profile a specific assignment. Similarly, an *extensive-form randomized mechanism* is a function $M : \mathcal{R}^N \to \Delta(\mathcal{S})$, which maps each profile to a lottery over assignments. However, in most cases, our primary concern is the outcome itself rather than the process by which it is generated. Therefore, we introduce the following definition.

Definition 2.1 (Normal-form randomized mechanism)**.** A *normal-form randomized mechanism* is a function $f : \mathcal{R}^N \to [0,1]^{H \times N}$ that directly maps each preference profile $\mathbf{P}$ to a probability matrix. For every house-agent pair $(h, i) \in H \times N$, the entry $f(\mathbf{P})_{h,i}$ denotes the probability that agent i receives house h.

For each $h \in H$, denote by $f(\mathbf{P})_h$ the row vector of $f(\mathbf{P})$ corresponding to h. Similarly, for each $i \in N$, denote by $f(\mathbf{P})_i$ the column vector of $f(\mathbf{P})$ corresponding to i. With a slight abuse of notation, we identify $f(\mathbf{P})_i$ with the probability distribution that assigns to agent i each house h with probability $f(\mathbf{P})_{h,i}$ and the null object with probability $1 - \sum_{h \in H} f(\mathbf{P})_{h,i}$.

We also define the normal form of an extensive-form randomized mechanism $M : \mathcal{R}^N \to \Delta(\mathcal{S})$ to be the normal-form randomized mechanism $f_M : \mathcal{R}^N \to [0,1]^{H \times N}$ that, for every preference profile $\mathbf{P}$, yields the same assignment probabilities as those induced by M. Specifically, for every $\mathbf{P} \in \mathcal{R}^N$, $i \in N$ and $h \in H$,

$$f_M(\mathbf{P})_{h,i} \coloneqq \sum_{s \in \mathcal{S} : s(i) = h} M(\mathbf{P})(s).$$

We say that two randomized mechanisms are *welfare equivalent* if they have the same normal form. Unless stated otherwise, we will use the term *mechanism* to refer to a randomized mechanism in its normal form.

Using the definitions above, we now introduce the *Random Serial Dictatorship (RSD)* mechanism and the main axioms used throughout the paper. To define RSD in its extensive form, we begin with the deterministic mechanisms that generate the assignments in the support of $\mathrm{RSD}\left(\mathbf{P}\right)$ for each $\mathbf{P} \in \mathcal{R}^N$.

Given a permutation $\sigma \in S_n$, the *Serial Dictatorship* mechanism SD_σ is the deterministic mechanism in which agents are ordered according to σ and each, in turn, selects their most preferred available house (i.e., one not chosen by any earlier agent in the sequence), and if $n > m$, the last $n - m$ agents get the null object.

Definition 2.2 (RSD)**.** The *Random Serial Dictatorship* mechanism, denoted RSD, is the randomized mechanism that samples a permutation $\sigma \in S_n$ uniformly at random and applies the corresponding serial dictatorship mechanism SD_σ. Thus, in its extensive form,

$$\mathrm{RSD}\left(\mathbf{P}\right) \coloneqq \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{SD}_\sigma \left(\mathbf{P}\right).$$

An assignment $s \in \mathcal{S}$ is *(Pareto) efficient* with respect to a profile $\mathbf{P} \in \mathcal{R}^N$ if there is no other assignment $s' \in \mathcal{S}$ such that some agent i strictly prefers $s'\left(i\right)$ to $s\left(i\right)$ without another agent j strictly preferring $s\left(j\right)$ to $s'\left(j\right)$. In other words, whenever there exists an agent i who strictly prefers $s'\left(i\right)$ over $s\left(i\right)$, there must also be at least one agent j who strictly prefers $s\left(j\right)$ over $s'\left(j\right)$.

A deterministic mechanism is *efficient* if it produces an efficient assignment for every profile. Similarly, an extensive form mechanism is *ex-post efficient* if, for every profile, its output distribution is supported entirely on efficient assignments with respect to that profile.

Given our primary focus on mechanisms in normal form, we define efficiency for them as follows:

Definition 2.3 (ExPE)**.** A normal form mechanism f satisfies *Ex-Post Pareto Efficiency (ExPE)* if there exists an ex-post efficient extensive-form mechanism M such that $f_M = f$.

The following fairness axiom reflects the idea that agents who are identical in all relevant aspects should be treated identically.

Definition 2.4 (ETE)**.** A mechanism $f : \mathcal{R}^N \to [0,1]^{H \times N}$ satisfies *Equal Treatment of Equals (ETE)* if, for every preference profile and for every pair of agents with identical preferences in that profile, the mechanism assigns them the same distribution over objects. Formally, for every $\mathbf{P} \in \mathcal{R}^N$ and every pair of agents $i, i' \in N$, if $P_i = P_{i'}$, then $f\left(\mathbf{P}\right)_i = f\left(\mathbf{P}\right)_{i'}$.

Given a profile $\mathbf{P} \in \mathcal{R}^N$ and a preference order $R_i \in \mathcal{R}$, we denote by $\left(\mathbf{P}_{-i}, R_i\right) \in \mathcal{R}^N$ the profile obtained by replacing agent i's preference in $\mathbf{P}$ with R_i, while keeping all other agents' preferences unchanged.

Definition 2.5 (SP)**.** A mechanism $f : \mathcal{R}^N \to [0,1]^{H \times N}$ is *strategy-proof (SP)* if for every profile $\mathbf{P} \in \mathcal{R}^N$, every agent $i \in N$, and every $R_i \in \mathcal{R}$, the following holds:

$$f\left(\mathbf{P}\right)_i \succeq_{P_i} f\left(\mathbf{P}_{-i}, R_i\right)_i.$$

In other words, each agent weakly prefers the distribution over the objects they receive when reporting their true preferences to any distribution they could obtain by misreporting unilaterally.

Having defined the main axioms, we next lay out notational conventions for agent and house renamings, then formalize associated invariance properties. We denote by Π the set of all permutations of N, that is, all bijections $N \to N$. Elements of Π will always be interpreted as *renamings of the agents*. By contrast, S_n refers to orderings of the agents. Similarly, we denote by Γ the set of all permutations of H, which we interpret as *renamings of the houses*.

Notation 2.1. Let $\mathbf{P} \in \mathcal{R}^N$, $\pi \in \Pi$ and $\tau \in \Gamma$. We write $(\pi, \tau)(\mathbf{P})$ for the preference profile obtained from $\mathbf{P}$ by renaming the agents according to π and the houses according to τ. Formally, if $\mathbf{Q} \coloneqq (\pi, \tau)(\mathbf{P})$, then for each $i \in N$ and $h, h' \in H$,

$$hQ_ih' \iff \tau^{-1}(h) P_{\pi^{-1}(i)} \tau^{-1}(h').$$

Moreover, we set $\pi(\mathbf{P}) \coloneqq (\pi, \mathrm{id})(\mathbf{P})$ and $\tau(\mathbf{P}) \coloneqq (\mathrm{id}, \tau)(\mathbf{P})$.

Notation 2.2. Let f be a mechanism, $\pi \in \Pi$, and $\tau \in \Gamma$. We write $(\pi, \tau)(f)$ for the mechanism obtained from f by renaming the agents according to π and the houses according to τ. Formally, for every $\mathbf{P} \in \mathcal{R}^N$, $h \in H$, and $i \in N$,

$$(\pi, \tau)(f)(\mathbf{P})_{h,i} \coloneqq f((\pi, \tau)(\mathbf{P}))_{\tau(h),\pi(i)}.$$

Moreover, we set $\pi(f) \coloneqq (\pi, \mathrm{id})(f)$ and $\tau(f) \coloneqq (\mathrm{id}, \tau)(f)$.

We now introduce two central invariance properties of mechanisms.

Definition 2.6 (Anonymity)**.** A mechanism f is *anonymous* if it is invariant under renamings of agents; that is, $\pi(f) = f$ for every $\pi \in \Pi$.

Definition 2.7 (Neutrality)**.** A mechanism f is *neutral* if it is invariant under renamings of houses; that is, $\tau(f) = f$ for every $\tau \in \Gamma$.

Remark 1 (Anonymity $\Rightarrow$ ETE). If two agents submit identical preferences, then any anonymous mechanism assigns them identical probabilistic assignments. Indeed, let f be anonymous and let $\mathbf{P}$ satisfy $P_i = P_j$. Let π be the transposition of i and j. Since $\pi(\mathbf{P}) = \mathbf{P}$ and $\pi(f) = f$, we obtain

$$f(\mathbf{P})_i = \pi(f)(\mathbf{P})_i = f(\pi(\mathbf{P}))_{\pi(i)} = f(\mathbf{P})_j,$$

so agents i and j receive the same probabilistic assignment.

2.2 The role of efficiency in characterizing outcomes

The following lemma, which is known in the literature (see, e.g., [8] for the balanced case where $n = m$), characterizes the set of efficient assignments for a given preference profile. It states that every efficient assignment arises from a serial dictatorship mechanism. Although this result is known, we include a proof for the convenience of the reader.

Lemma 2.1. Let $\mathbf{P} \in \mathcal{R}^N$ be a preference profile. Then the set of all efficient assignments with respect to $\mathbf{P}$ is

$$\{\mathrm{SD}_\sigma(\mathbf{P}) \mid \sigma \in S_n\}.$$

Proof. We first show that $\mathrm{SD}_\sigma(\mathbf{P})$ is an efficient assignment with respect to $\mathbf{P}$ for every $\sigma \in S_n$. Suppose, for contradiction, that this is not the case. Then there exists an assignment $s \in \mathcal{S}$ such that each agent weakly prefers their object in s to their object in $\mathrm{SD}_\sigma(\mathbf{P})$, and at least one agent strictly prefers their object in s. Denote $A \coloneqq \{i \in N \mid s(i) \neq \mathrm{SD}_\sigma(\mathbf{P})(i)\}$. Then A is nonempty, and for every $i \in A$, we have $s(i)\, P_i^+ \mathrm{SD}_\sigma(\mathbf{P})(i)$.

Let $j \in A$ be the agent who appears first in the ordering σ among all agents in A. When j's turn arrives in the execution of $\mathrm{SD}_\sigma(\mathbf{P})$, he selects $\mathrm{SD}_\sigma(\mathbf{P})(j)$. Since $s(j)\, P_j^+ \mathrm{SD}_\sigma(\mathbf{P})(j)$, it must be that $s(j)$ was already chosen by some earlier agent. By the minimality of j in A, this agent must lie in $N \setminus A$; that is, there exists $i \in N \setminus A$ such that $\mathrm{SD}_\sigma(\mathbf{P})(i) = s(j)$. But then $s(i) = s(j)$, which contradicts the feasibility of s, unless $s(j) = \varnothing$. However, this contradicts the assumption that j strictly prefers $s(j)$ over $\mathrm{SD}_\sigma(\mathbf{P})(j)$. This contradiction shows that $\mathrm{SD}_\sigma(\mathbf{P})$ must be efficient.

To show that these are the only efficient assignments, let $s \in \mathcal{S}$ be an assignment that is efficient with respect to $\mathbf{P}$. We will construct an ordering $\sigma \in S_n$ such that $s = \mathrm{SD}_\sigma(\mathbf{P})$.

We begin by showing that the efficiency of s implies that at least one agent receives his top choice under s. Assume, for contradiction, that no agent receives his top choice. For each $i \in N$, let $h_i \in H$ denote his top choice under $\mathbf{P}$. Define a directed graph whose vertices are the agents, and draw a directed edge from agent i to agent j if and only if $s(j) = h_i$; that is, if agent j holds the house that agent i most prefers.

Since the number of agents is finite, this directed graph is finite and must contain either a directed cycle or a sink. In the case of a directed cycle, the agents involved in the cycle can reassign the houses among themselves along the cycle, so that each receives a strictly preferred house compared to s, contradicting the efficiency of s. In the case of a sink, there exists an agent $i \in N$ such that h_i is not assigned to anyone. In that case, agent i could be reassigned to h_i, which he strictly prefers over $s(i)$, without affecting the assignment of other agents, again contradicting the efficiency of s. Therefore, at least one agent must receive his top choice under s.

Let $\sigma(1)$ be such an agent. Remove agent $\sigma(1)$ and the house assigned to him from the problem, and consider the induced assignment and profile on the remaining agents and houses. Since s is efficient, the restricted assignment remains efficient with respect to the restricted profile. By the same reasoning as above, there exists an agent among the remaining ones who receives his top choice in the restricted problem. Let this agent be $\sigma(2)$, and repeat the process inductively. If no houses remain, the remaining agents can be ordered arbitrarily. In this way, we construct an ordering $\sigma \in S_n$ such that $s = \mathrm{SD}_\sigma(\mathbf{P})$, as desired. ■

We conclude from the above lemma that any mechanism f satisfying ExPE must, for each profile $\mathbf{P} \in \mathcal{R}^N$, produce an assignment matrix that lies in the convex hull of the assignment matrices corresponding to the assignments $\{\mathrm{SD}_\sigma(\mathbf{P}) \mid \sigma \in S_n\}$. In particular, whenever a given entry is zero in all such assignment matrices, it must be zero also in $f(\mathbf{P})$. This leads to the following weaker version of ExPE (which has appeared in the literature, particularly in the context of attempts to characterize RSD; see, e.g., [9, 17])

Definition 2.8 (Support efficiency)**.** A mechanism f satisfies *support efficiency* if for every preference profile $\mathbf{P} \in \mathcal{R}^N$, every house $h \in H$, and every agent $i \in N$, the following holds:
If $\mathrm{SD}_\sigma(\mathbf{P})(i) \neq h$ for every ordering $\sigma \in S_n$, then

$$f(\mathbf{P})_{h,i} = 0.$$

That is, if agent i is never assigned house h under the SD mechanism with any ordering of the agents at profile $\mathbf{P}$, then the mechanism f must assign probability zero to the pair (h,i) at profile $\mathbf{P}$.

Alongside support efficiency, we will also use the following weak implication of ExPE.

Definition 2.9 (Full assignment property)**.** A mechanism f satisfies the *full assignment property* if, for every preference profile $\mathbf{P}$, the following hold:

1. If an agent i is assigned a house in every SD mechanism at $\mathbf{P}$, then i is assigned a house with probability 1 under $f(\mathbf{P})$.
2. If a house h is assigned to an agent in every SD mechanism at $\mathbf{P}$, then h is assigned to an agent with probability 1 under $f(\mathbf{P})$.

In the balanced case $m = n$, every SD outcome assigns a house to each agent and allocates each house. Hence, in this case, Definition 2.9 is equivalent to requiring that, for every profile $\mathbf{P}$, the assignment matrix $f(\mathbf{P})$ is bi-stochastic, that is, all row and column sums equal 1.

Remark 2. All the uniqueness proofs in Section 4 use only support efficiency and the full assignment property instead of the full ExPE axiom.

2.3 A reformulation of Strategy-Proofness

The following lemma provides a standard local characterization of strategy-proofness: it shows that SP is equivalent to requiring that an agent cannot benefit from swapping two adjacent houses in their preference. This form is well known in the literature (see, e.g., [14] for the case where $n \leq m$), but for the convenience of the reader, we include a proof.

Lemma 2.2. A mechanism f satisfies SP if and only if, for every preference profile $\mathbf{P} = (P_1, \ldots, P_n) \in \mathcal{R}^N$, every agent $i \in N$, and every pair of adjacent houses h', h'' in P_i, where $h'' P_i^+ h'$, the following conditions hold:

$$\begin{aligned} f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h',i} &\geq f(\mathbf{P})_{h',i}, \\ \forall h \in H \setminus \{h', h''\} : f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h,i} &= f(\mathbf{P})_{h,i}, \\ f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h',i} - f(\mathbf{P})_{h',i} &= f(\mathbf{P})_{h'',i} - f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h'',i}. \end{aligned}$$

Here, $P_i^{h',h''}$ denotes the preference obtained from P_i by swapping the adjacent houses h' and h''.

Proof. Assume first that f satisfies SP. Let $\mathbf{P} \in \mathcal{R}^N$, let $i \in N$, and let h', h'' be adjacent houses in P_i, with $h'' P_i^+ h'$. Denote $\mathbf{Q} := \left(\mathbf{P}_{-i}, P_i^{h',h''}\right)$, so that $\mathbf{P} = \left(\mathbf{Q}_{-i}, Q_i^{h'',h'}\right)$. Then, by the definition of SP, we have:

$$\sum_{h \in C_{Q_i}(h')} f(\mathbf{Q})_{h,i} \geq \sum_{h \in C_{Q_i}(h')} f(\mathbf{P})_{h,i}.$$

If h' is the top-ranked house in Q_i, then $C_{Q_i}(h') = \{h'\}$, and the desired inequality follows immediately. Otherwise, let h_0 denote the house immediately above h' in Q_i. Then, applying SP in the reverse direction, we obtain:

$$\sum_{h \in C_{P_i}(h_0)} f(\mathbf{P})_{h,i} \geq \sum_{h \in C_{P_i}(h_0)} f(\mathbf{Q})_{h,i}.$$

Note that $C_{P_i}(h_0) = C_{Q_i}(h') \setminus \{h'\}$. Subtracting the two inequalities above, we obtain $f(\mathbf{Q})_{h',i} \geq f(\mathbf{P})_{h',i}$ which establishes the first condition.

Now let $h \in H \setminus \{h', h''\}$. By SP, we have:

$$\sum_{a \in C_{P_i}(h)} f(\mathbf{P})_{a,i} \geq \sum_{a \in C_{P_i}(h)} f(\mathbf{Q})_{a,i} \text{ and } \sum_{a \in C_{Q_i}(h)} f(\mathbf{Q})_{a,i} \geq \sum_{a \in C_{Q_i}(h)} f(\mathbf{P})_{a,i}.$$

Since $C_{P_i}(h) = C_{Q_i}(h)$, all these sums must be equal. In particular,

$$\sum_{a \in C_{P_i}(h)} f(\mathbf{P})_{a,i} = \sum_{a \in C_{Q_i}(h)} f(\mathbf{Q})_{a,i}.$$

If h is the top-ranked house, this directly implies $f(\mathbf{Q})_{h,i} = f(\mathbf{P})_{h,i}$. Otherwise, assume that h is not the house immediately below h' in P_i (if such a house exists). Let a_h denote the house ranked directly above h in P_i. Then we have:

$$f(\mathbf{Q})_{h,i} = \sum_{a \in C_{Q_i}(h)} f(\mathbf{Q})_{a,i} - \sum_{a \in C_{Q_i}(a_h)} f(\mathbf{Q})_{a,i} = \sum_{a \in C_{P_i}(h)} f(\mathbf{P})_{a,i} - \sum_{a \in C_{P_i}(a_h)} f(\mathbf{P})_{a,i} = f(\mathbf{P})_{h,i},$$

where the equalities follow from the fact that cumulative assignment probabilities are equal for all $h \notin \{h', h''\}$. We have thus established the second condition for all relevant houses, except the house immediately below h' in P_i, if such a house exists.

We now move on to verify the third condition of the lemma. By SP, we have the following two inequalities:

$$\sum_{a \in C_{P_i}(h')} f(\mathbf{P})_{a,i} \geq \sum_{a \in C_{P_i}(h')} f(\mathbf{Q})_{a,i} \text{ and } \sum_{a \in C_{Q_i}(h'')} f(\mathbf{Q})_{a,i} \geq \sum_{a \in C_{Q_i}(h'')} f(\mathbf{P})_{a,i}.$$

Since $C_{P_i}(h') = C_{Q_i}(h'')$, both inequalities must in fact be equalities. Moreover,

$$\sum_{a \in C_{P_i}(h')} f(\mathbf{P})_{a,i} = \sum_{a \in C_{Q_i}(h'')} f(\mathbf{Q})_{a,i}\,.$$

If h'' is the top-ranked house in P_i, then rearranging proves the third condition. Otherwise, let a' be the house ranked immediately above h'' in P_i, and note that:

$$\begin{aligned} f(\mathbf{P})_{h'',i} + f(\mathbf{P})_{h',i} &= \sum_{a \in C_{P_i}(h')} f(\mathbf{P})_{a,i} - \sum_{a \in C_{P_i}(a')} f(\mathbf{P})_{a,i} \\ &= \sum_{a \in C_{Q_i}(h'')} f(\mathbf{Q})_{a,i} - \sum_{a \in C_{Q_i}(a')} f(\mathbf{Q})_{a,i} \\ &= f(\mathbf{Q})_{h',i} + f(\mathbf{Q})_{h'',i}\,, \end{aligned}$$

which establishes the third condition.

Let b be the house ranked immediately below h' in P_i, if such a house exists. Then:

$$f(\mathbf{P})_{b,i} = \sum_{a \in C_{P_i}(b)} f(\mathbf{P})_{a,i} - \sum_{a \in C_{P_i}(h')} f(\mathbf{P})_{a,i} = \sum_{a \in C_{Q_i}(b)} f(\mathbf{Q})_{a,i} - \sum_{a \in C_{Q_i}(h'')} f(\mathbf{Q})_{a,i} = f(\mathbf{Q})_{b,i}\,,$$

completing the verification of the second condition for this remaining case.

Now suppose the three conditions stated in the lemma hold for every preference profile and every pair of adjacent houses in an agent's ranking. We will show that f satisfies SP. Let $\mathbf{P} \in \mathcal{R}^N$, $i \in N$ and let $R_i \in \mathcal{R}$ be an alternative preference order for agent i. Denote by k the minimal number of adjacent swaps required to transform P_i into R_i. We prove by induction on k that $f(\mathbf{P})_i \succeq_{P_i} f(\mathbf{P}_{-i}, R_i)$. The base case $k = 0$ is immediate since $R_i = P_i$. Assume the statement holds for all preference orders reachable from P_i by fewer than k adjacent swaps. Let h', h'' be adjacent houses in R_i such that $h'' P_i^+ h'$ but $h' R_i^+ h''$, and let R_i' be obtained from R_i by swapping h' and h''. Since the number of swaps needed to go from P_i to R_i' is less than k, the induction hypothesis gives $f(\mathbf{P})_i \succeq_{P_i} f(\mathbf{P}_{-i}, R_i')$. Assume, for contradiction, that $f(\mathbf{P})_i \not\succeq_{P_i} f(\mathbf{P}_{-i}, R_i)$. Then there exists some house $h_0 \in H$ such that:

$$\sum_{h \in C_{P_i}(h_0)} f(\mathbf{P})_{h,i} < \sum_{h \in C_{P_i}(h_0)} f(\mathbf{P}_{-i}, R_i)_{h,i}\,.$$

However, by the induction hypothesis, we have

$$\sum_{h \in C_{P_i}(h_0)} f(\mathbf{P})_{h,i} \geq \sum_{h \in C_{P_i}(h_0)} f(\mathbf{P}_{-i}, R_i')_{h,i}\,,$$

which implies that

$$\sum_{h \in C_{P_i}(h_0)} f(\mathbf{P}_{-i}, R_i')_{h,i} < \sum_{h \in C_{P_i}(h_0)} f(\mathbf{P}_{-i}, R_i)_{h,i}\,.$$

By the three conditions of the lemma, this strict inequality can hold only if $h' \in C_{P_i}(h_0)$ and $h'' \notin C_{P_i}(h_0)$, which contradicts the fact that $h'' P_i^+ h'$. Therefore, no such h_0 exists, and f

satisfies SP. ■

The next definition isolates the two invariance requirements obtained from Lemma 2.2 by omitting the inequality; their conjunction is therefore weaker than SP.

Definition 2.10 (Upper and lower invariance)**.** A mechanism f satisfies *upper invariance* if, for every preference profile $\mathbf{P} \in \mathcal{R}^N$, every agent $i \in N$, and every pair of adjacent houses h', h'' in P_i, where $h'' P_i^+ h'$, for every $h \in H$ such that $h P_i^+ h''$ we have

$$f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h,i} = f\left(\mathbf{P}\right)_{h,i}.$$

Here, $P_i^{h',h''}$ is obtained from P_i by swapping the adjacent houses h' and h'', as in Lemma 2.2.

Similarly, f satisfies *lower invariance* if the above equality holds for every $h \in H$ such that $h' P_i^+ h$, and, in addition, the probability that agent i receives nothing does not change under this swap.

Remark 3. All the uniqueness proofs in Section 4 use only upper invariance and lower invariance rather than the full SP axiom. Moreover, in Section 3 (the case $m = 2$), Proposition 3.1 remains valid when SP is replaced by lower invariance.

Remark 4. If f satisfies upper and lower invariance and $f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h'',i} = f\left(\mathbf{P}\right)_{h'',i}$, then $f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h',i} = f\left(\mathbf{P}\right)_{h',i}$ as well. Indeed, upper and lower invariance imply that agent i's probability of receiving any house other than h' or h'' is unchanged by the swap, and lower invariance also ensures that the probability that agent i receives nothing is unchanged. By complementing probabilities, it follows that the total probability that agent i receives either h' or h'' is the same under $\mathbf{P}$ and under $\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)$. Combining this with the given equality for h'' yields the desired equality for h'. Note that when f satisfies ExPE (or even just support efficiency), we can use this fact when i cannot receive h'' under any efficient assignment at $\mathbf{P}$, because in that case we would have $f\left(\mathbf{P}_{-i}, P_i^{h',h''}\right)_{h'',i} = f\left(\mathbf{P}\right)_{h'',i} = 0$.

2.4 Equal Treatment for All

A useful property that follows from the ExPE, ETE, and SP axioms concerns how a mechanism must distribute a given house among agents with similar incentives. Intuitively, if a house h is assigned with positive probability to several agents, and these agents have identical binary preferences between h and other houses (that is, for each other house $h' \neq h$, they either all prefer h' to h or all prefer h to h'), then the mechanism must treat them symmetrically with respect to h. That is, each of them must receive h with equal probability. We refer to this implication as *Equal Treatment for All (ETA).*

Proposition 2.1 (ETA)**.** Let f be a mechanism that satisfies ExPE, ETE and SP. Let $h \in H$ and $\mathbf{P} = (P_1, \ldots, P_n) \in \mathcal{R}^N$. Define $I_{\mathbf{P}} \coloneqq \{i \in N \mid \exists \sigma \in S_n : \mathrm{SD}_\sigma\left(\mathbf{P}\right)(i) = h\}$. Suppose that $\mathbf{P}$ is a profile such that, under every SD mechanism, the house h is assigned to some agent, and that $C_{P_i}(h) = C_{P_{i'}}(h)$ for all $i, i' \in I_{\mathbf{P}}$. Then $f\left(\mathbf{P}\right)_{h,i} = \frac{1}{|I_{\mathbf{P}}|}$ for every $i \in I_{\mathbf{P}}$.

Proof. By assumption, the set $I_{\mathbf{P}}$ is non-empty. Since f satisfies ExPE, it follows that $\sum_{i\in N} f(\mathbf{P})_{h,i} = 1$ and $f(\mathbf{P})_{h,j} = 0$ for all $j \in N \setminus I_{\mathbf{P}}$. Let $E_{\mathbf{P}} \subseteq I_{\mathbf{P}}$ be a largest subset of agents with identical preferences. The result is established via induction on $|I_{\mathbf{P}} \setminus E_{\mathbf{P}}|$.

The base case $E_{\mathbf{P}} = I_{\mathbf{P}}$ follows directly from the ETE property of f. For the inductive step, let $P \in \mathcal{R}$ denote the common preference of agents in $E_{\mathbf{P}}$. For each $i' \in I_{\mathbf{P}} \setminus E_{\mathbf{P}}$, consider the modified profile $\mathbf{P}^{i'} := (\mathbf{P}_{-i'}, P)$. Assuming $I_{\mathbf{P}^{i'}} = I_{\mathbf{P}}$, note that $E_{\mathbf{P}^{i'}} = E_{\mathbf{P}} \cup \{i'\}$, so $|I_{\mathbf{P}^{i'}} \setminus E_{\mathbf{P}^{i'}}| < |I_{\mathbf{P}} \setminus E_{\mathbf{P}}|$. By the induction hypothesis, $f\left(\mathbf{P}^{i'}\right)_{h,i} = \frac{1}{|I_{\mathbf{P}^{i'}}|} = \frac{1}{|I_{\mathbf{P}}|}$ for every $i \in I_{\mathbf{P}^{i'}} = I_{\mathbf{P}}$, and in particular for $i = i'$. Since $P_{i'}$ can be transformed to P via adjacent swaps not involving h, SP implies $f(\mathbf{P})_{h,i'} = \frac{1}{|I_{\mathbf{P}}|}$ for each $i' \in I_{\mathbf{P}} \setminus E_{\mathbf{P}}$. To extend the equality to all agents $i \in E_{\mathbf{P}}$, we use the two facts derived from ExPE mentioned earlier, together with the already established equality for agents in $I_{\mathbf{P}} \setminus E_{\mathbf{P}}$, and finally ETE, which together imply $f(\mathbf{P})_{h,i} = \frac{1}{|I_{\mathbf{P}}|}$ for all $i \in E_{\mathbf{P}}$. It is left to show that $I_{\mathbf{P}^{i'}} = I_{\mathbf{P}}$ holds for every $i' \in I_{\mathbf{P}} \setminus E_{\mathbf{P}}$. To establish this, we prove the following lemma, which concludes the proof of the proposition.

Lemma 2.3. Let $h \in H$ and let $\mathbf{P} \in \mathcal{R}^N$ be a profile satisfying the requirements of the proposition. Let $\mathbf{P}' \in \mathcal{R}^N$ be a profile that differs from $\mathbf{P}$ only in the preferences of agents in $I_{\mathbf{P}}$ (defined as before). If $C_{P'_i}(h) = C_{P_i}(h)$ for every agent $i \in I_{\mathbf{P}}$, then $I_{\mathbf{P}'} = I_{\mathbf{P}}$.

Proof. Let $I := I_{\mathbf{P}}$, $J := N \setminus I$, and $T := C_{P_i}(h) \setminus \{h\}$ for any $i \in I$ (by hypothesis, T is identical for all $i \in I$). Denote $t := |T|$. We first show that $t < |I|$.

Suppose, for contradiction, that $t \geq |I|$. Consider the assignment $\mathrm{SD}_\sigma(\mathbf{P})$ induced by a permutation σ that places all agents in I before those in J. Since each agent in I strictly prefers every house in T to h and $t \geq |I|$, each agent in I selects a house from T. Consequently, house h remains unassigned after processing I. By the assumption on $\mathbf{P}$, the house h must be assigned to some agent in J. This contradicts the definition of I, as no agent in J can receive h under any SD mechanism. Thus, $t < |I|$.

We establish $I \subseteq I_{\mathbf{P}'}$. Let $i \in I$. Consider a permutation σ that orders all agents in I in the first $|I|$ positions, with i at position $t+1$. Since $C_{P'_{i'}}(h) = T \cup \{h\}$ for all $i' \in I$, under $\mathrm{SD}_\sigma(\mathbf{P}')$, the first t agents (each in I) select distinct houses from T, exhausting T. Consequently, agent i selects h. Thus, $i \in I_{\mathbf{P}'}$.

For the converse inclusion, define Σ as the set of permutations that assign t agents from I to the first t positions, then all agents in J to the next $|J|$ positions, and the remaining agents from I to the subsequent positions. Formally,

$$\Sigma := \{\sigma \in S_n \mid \sigma(J) = [t + |J|] \setminus [t]\}.$$

Note that for every $\sigma \in \Sigma$ and every $j \in J$, $(\mathrm{SD}_\sigma(\mathbf{P})(j))\, P_j^{+} h$. This is because, under such a permutation, the first t positions are occupied by agents from I, who select all houses in T, and the agents in J that come before j do not take h. Thus, h is available when it is j's turn, but since j does not receive h, it must be that j chooses a house that he strictly prefers to h.

We demonstrate that the same holds for $\mathbf{P}'$ (i.e., $(\mathrm{SD}_\sigma(\mathbf{P}')(j))\, P_j'^{+} h$). Given that $P'_j = P_j$, it suffices to establish $\mathrm{SD}_\sigma(\mathbf{P}')(j) = \mathrm{SD}_\sigma(\mathbf{P})(j)$. This equality follows because, in both profiles, the first t agents from I select all houses in T. By induction on the ordering of agents in J, we conclude that each agent in J chooses the same house under $\mathbf{P}'$ as under $\mathbf{P}$.

Let $\tau \in S_n$. We aim to demonstrate that $\mathrm{SD}_\tau(\mathbf{P}')(j) \neq h$ for all $j \in J$. Let σ_τ be the permutation in Σ preserving the relative order of agents within I and within J from τ. Specifically, for any $i, i' \in I$, agent i precedes agent i' in σ_τ if and only if the same holds in τ, and analogously for agents in J.

For every agent $j \in J$, define $h_j := \mathrm{SD}_{\sigma_\tau}(\mathbf{P}')(j)$. As established previously, $h_j P_j'^{+} h$. Moreover, since $\sigma_\tau \in \Sigma$, it follows that $h_j \notin T$ because the first t agents from I exhaust T.

Assume by way of contradiction that there exists an agent $j_0 \in J$ such that $\mathrm{SD}_\tau(\mathbf{P}')(j_0) = h$. Let $j' \in J$ be the first agent in the ordering τ for which, during the execution of $\mathrm{SD}_\tau(\mathbf{P}')$, the house $h_{j'}$ was unavailable for selection by j', but the house h was available. Such an agent exists because the condition holds for j_0.

By the definition of j', for every agent $i \in I$ preceding j' with respect to τ, we have $\mathrm{SD}_\tau(\mathbf{P}')(i) \in T$ (because otherwise one such agent would have selected h, contradicting h's availability for j') and thus $\mathrm{SD}_\tau(\mathbf{P}')(i) \neq h_{j'}$ for every such i. Since $h_{j'}$ is unavailable for selection by j' during the execution of $\mathrm{SD}_\tau(\mathbf{P}')$, there exists an agent $j'' \in J$ preceding j' such that $\mathrm{SD}_\tau(\mathbf{P}')(j'') = h_{j'}$. Note that h is available for j'' during $\mathrm{SD}_\tau(\mathbf{P}')$ (as j'' precedes j'). Because σ_τ preserves the relative order of agents within J from τ, j'' precedes j' in σ_τ, implying $h_{j''} P_{j''}'^{+} h_{j'}$ (since both houses were available to j'' during $\mathrm{SD}_{\sigma_\tau}(\mathbf{P}')$ and he selected $h_{j''}$). However, during $\mathrm{SD}_\tau(\mathbf{P}')$, j'' selected $h_{j'}$, indicating $h_{j''}$ was unavailable to him - contradicting the minimality of j' in τ. Consequently, $\mathrm{SD}_\tau(\mathbf{P}')(j) \neq h$ for all $j \in J$ and $\tau \in S_n$, which implies $I_{\mathbf{P}'} \subseteq I$, concluding the proof. ■

■

2.5 Near-unanimity forces uniqueness

To better understand the implications of the axioms ExPE, ETE, and SP, we analyze a structured class of preference profiles in which agents exhibit near-unanimous agreement on the relative ranking of every pair of houses. In such highly structured settings, the combined force of the axioms leaves essentially no room for flexibility: the outcome is uniquely pinned down. The following lemma formalizes this observation by proving uniqueness of the assignment matrix under these conditions.

Lemma 2.4. Assume $n \geq \max\{m, 4\}$. Let $\mathbf{P} \in \mathcal{R}^N$ be a preference profile such that for every pair of distinct houses $\{h_1, h_2\} \subseteq H$, all but possibly one agent prefer the same house between h_1 and h_2. Then, the assignment matrix of $\mathbf{P}$ is uniquely determined across all mechanisms satisfying ExPE, ETE, and SP.

Proof. We prove the lemma by induction on the number of unordered pairs $\{h_1, h_2\} \subseteq H$ for which the agents do not unanimously prefer one house over the other. In the base case, all agents agree on the binary preference between every pair of houses. In other words, they have identical rankings.

Since $n \geq m$, ExPE ensures that each house is fully allocated, and ETE requires that each house is shared equally among the agents. Hence, the assignment matrix is uniquely determined.

For the induction step, let $\mathbf{P} \in \mathcal{R}^N$ be a preference profile with $d > 0$ unordered pairs of houses for which the agents do not unanimously prefer one house over the other. Let these d house pairs be $\{h_1, h'_1\}, \ldots, \{h_d, h'_d\}$. For each $\ell \in [d]$, let $a_\ell \in N$ be the agent whose binary preference over the pair $\{h_\ell, h'_\ell\}$ differs from that of all other agents. Note that even if $\ell_1 \neq \ell_2$, it is possible that a_{ℓ_1} and a_{ℓ_2} refer to the same agent.

We first claim that for every $\ell \in [d]$, there exists a pair of adjacent houses in the ranking of agent a_ℓ such that this agent prefers one house over the other, while all other agents prefer the other house in that pair. To see this, consider a pair $\{h, h'\}$ for which a_ℓ is the agent whose binary preference differs from the rest. If h and h' are not adjacent in the ranking of a_ℓ, then there exists a house h'' that appears between them in that ranking. In that case, a_ℓ must disagree with every other agent on the binary preference of at least one of the pairs $\{h, h''\}$ or $\{h', h''\}$. Since $n \geq 4$, for at least one of these two pairs there must be two agents whose binary preference differs from that of a_ℓ. Without loss of generality, suppose this pair is $\{h, h''\}$. Since at most one agent can disagree with the rest on any given pair, it follows that all other agents share the same preference over $\{h, h''\}$, and that this preference differs from that of a_ℓ. Repeating this process eventually leads to a pair of adjacent houses in the ranking of a_ℓ, for which his binary preference over that pair differs from that of all the other agents.

Let $\{h, h'\}$ be such a pair, and let $\mathbf{P}_\ell$ be the profile obtained by swapping the positions of h and h' in the ranking of agent a_ℓ. In this modified profile, there are only $d - 1$ house pairs for which there is no consensus among the agents regarding their binary ranking. By the induction hypothesis, the assignment matrix for $\mathbf{P}_\ell$ is uniquely determined by the axioms.

By SP, the probabilities that a_ℓ receives any house $h'' \notin \{h, h'\}$ must remain the same in both $\mathbf{P}$ and $\mathbf{P}_\ell$. Since a_ℓ is the only agent whose preference over the pair $\{h, h'\}$ differs from the others, efficiency implies that under the profile $\mathbf{P}$, he cannot receive the house he ranks lower between the two. It then follows from SP that his probability of receiving the house he ranks higher is also determined. Therefore, the probabilities assigned to each agent in the set $\{a_1, \ldots, a_d\}$ are uniquely determined by the axioms in the profile $\mathbf{P}$.

Now, let $N' \coloneqq N \setminus \{a_1, \ldots, a_d\}$. By definition of the agents $a_1, \ldots, a_d$, all agents in N' share the same ranking in $\mathbf{P}$. This set may be empty, in which case the proof is complete. Otherwise, since $n \geq m$, the axiom ExPE guarantees that each house is fully allocated, and by ETE, all agents in N' must receive the same probability for each house. Therefore, the assignment probabilities for all agents in N' are uniquely determined in the profile $\mathbf{P}$, which completes the proof. ■

3 Characterization for $m = 2$

In the following section, we characterize the mechanisms satisfying ExPE, ETE, and SP for $n \geq 2$ and $m = 2$.

Denote the set of houses by $H \coloneqq \{a, b\}$. For each preference profile $\mathbf{Q} \in \mathcal{R}^N$ and house $h \in H$, let $N_{\mathbf{Q},h} \subseteq N$ be the subset of agents who rank h first. Define $n_{\mathbf{Q},h} \coloneqq |N_{\mathbf{Q},h}|$ to be the number of such agents.

We begin by establishing key facts that will allow us to interpret a mechanism satisfying the axioms as a function. By ExPE, each house is allocated with probability 1.

Furthermore, using induction on $n_{\mathbf{Q},a}$, we establish that each agent is assigned a house with probability $\frac{2}{n}$.

First, as each of the two houses is fully allocated, the total probability of being assigned a house across all agents is 2. Second, when all agents have identical preferences, ETE ensures that they must all receive the same probability of being assigned, which must therefore be $\frac{2}{n}$. This establishes the base case $n_{\mathbf{Q},a} = 0$, and the argument applies equally to the case $n_{\mathbf{Q},a} = n$. Now consider the case where $0 < n_{\mathbf{Q},a} < n$. Let $i \in N_{\mathbf{Q},a}$, and let $\mathbf{Q}'$ be the profile in which agent i ranks b above a, while the preferences of all other agents remain as in $\mathbf{Q}$. Note that $n_{\mathbf{Q}',a} < n_{\mathbf{Q},a}$, so by the induction hypothesis, every agent in $\mathbf{Q}'$ is assigned with probability $\frac{2}{n}$. Then, by SP, agent i must also be assigned with probability $\frac{2}{n}$ in the original profile $\mathbf{Q}$. Since there are only two types of preferences in $\mathbf{Q}$, and the total sum of the agents' assignment probabilities is 2, ETE and the complementarity of probabilities together imply that all agents in $\mathbf{Q}$ are assigned with probability $\frac{2}{n}$.

Consequently, by ETE and the complementarity of probabilities, it suffices to determine the probability that an agent in $N_{\mathbf{Q},a}$ receives house a. In other words, any mechanism satisfying the axioms is uniquely determined once this probability is specified.

Hence, we consider functions of the form $\varphi : 2^N \setminus \{\varnothing\} \to \left[0, \frac{2}{n}\right]$, and define the corresponding mechanism $f(\varphi)$ to be the mechanism that satisfies ETE and, for every $\mathbf{Q} \in \mathcal{R}^N$, assigns each agent in $N_{\mathbf{Q},a}$ to house a with probability $\varphi(N_{\mathbf{Q},a})$, allocates each house with probability 1, and assigns each agent with probability $\frac{2}{n}$. The empty set is excluded from the domain of φ, since assigning agents in $\varnothing$ to a with any probability is meaningless. Moreover, if $\varphi_1 \neq \varphi_2$, then $f(\varphi_1) \neq f(\varphi_2)$, ensuring that each such function induces a different mechanism.

Proposition 3.1. Let $n \geq 2$ and $m = 2$. The correspondence described above defines a bijection between the set of mechanisms satisfying ExPE, ETE and SP, and the set of functions:

$$\mathcal{F} := \left\{ \varphi : 2^N \setminus \{\varnothing\} \to \left[\frac{1}{n}, \frac{2}{n}\right] \mid \begin{array}{c} \forall R \in 2^N \setminus \{\varnothing\} : \varphi(R) \leq \frac{1}{|R|} \\ \forall i \in N : \varphi(\{i\}) = \frac{2}{n}, \varphi(N \setminus \{i\}) = \frac{1}{n-1} \end{array} \right\}$$

Proof. We first show that any mechanism f satisfying the axioms must correspond to a function in $\mathcal{F}$. First, observe that when $n_{\mathbf{Q},a} \in \{0, 1, n-1, n\}$, the probabilities in the profile are fully determined by ExPE and ETE. Specifically, this corresponds to

$$\varphi(\{i\}) = \frac{2}{n},\ \varphi(N \setminus \{i\}) = \frac{1}{n-1},\ \text{and } \varphi(N) = \frac{1}{n}.$$

The last condition is implicitly satisfied in the definition of $\mathcal{F}$ since the range of the functions is constrained to $\left[\frac{1}{n}, \frac{2}{n}\right]$, so since $\varphi(R) \leq \frac{1}{|R|}$ we obtain

$$\frac{1}{n} \leq \varphi(N) \leq \frac{1}{|N|} = \frac{1}{n}.$$

Thus, the constraint holds automatically.

Now, since $\varphi(R)$ represents the probability that an agent $i \in R = N_{\mathbf{Q},a}$ receives house a, we obtain the constraint $|R|\varphi(R) \leq 1$. Rearranging, this yields $\varphi(R) \leq \frac{1}{|R|}$.

It remains to show that $\varphi(R) \geq \frac{1}{n}$ for any $R \in 2^N \setminus \{\varnothing\}$. Let $R \in 2^N \setminus \{\varnothing\}$ and denote by $\mathbf{Q} \in \mathcal{R}^N$ the corresponding profile such that $R = N_{\mathbf{Q},a}$. When the profile is $\mathbf{Q}$, an agent in R receives a with probability $\varphi(R)$ and b with probability $\frac{2}{n} - \varphi(R)$. Thus, the probability that some agent in R receives a is $|R|\varphi(R)$ while the probability that some agent in R receives b is $|R|\left(\frac{2}{n} - \varphi(R)\right)$. Since the mechanism satisfies ExPE, it can be expressed as a convex combination of deterministic serial dictatorship mechanisms. In these mechanisms, whenever an agent in $R = N_{\mathbf{Q},a}$ receives b, another agent in R must receive a. This implies that the event of some agent in R receiving b (in the mechanism f when the profile is $\mathbf{Q}$) is contained in the event of some agent in R receiving a, which yields

$$|R|\left(\frac{2}{n} - \varphi(R)\right) \leq |R|\varphi(R)$$
$$\implies \varphi(R) \geq \frac{1}{n}.$$

Thus, we conclude that any mechanism satisfying the axioms corresponds to a function in $\mathcal{F}$.

Conversely, let $\varphi \in \mathcal{F}$. We will show that the corresponding mechanism $f(\varphi)$ satisfies the axioms. Since the correspondence was defined to preserve equal treatment among agents with identical preferences, it follows that $f(\varphi)$ satisfies ETE.

Now, we will show that $f(\varphi)$ satisfies SP in two steps: first, we will show that if $\varphi \in \mathcal{F}$ satisfies a certain constraint, then $f(\varphi)$ satisfies SP. Next, we will show that every $\varphi \in \mathcal{F}$ satisfies this constraint.

As a first step, for every $R \subsetneq N$, define

$$g_\varphi(R) \coloneqq \frac{1 - |R|\varphi(R)}{n - |R|},$$

with the convention that $|R|\varphi(R) \coloneqq 0$ when $R = \varnothing$. Note that since $\varphi(R)$ represents the probability that an agent who prefers a gets a, $g_\varphi(R)$ represents the probability that an agent who prefers b gets a. We will show that if φ satisfies the constraint $\varphi(R) \geq g_\varphi(R \setminus \{i\})$ for every nonempty $R \subseteq N$ and for every $i \in R$, then $f(\varphi)$ satisfies SP.

Indeed, let $i \in N$. We will show that i cannot gain from manipulation. Since there are only two houses, and every agent receives some house with a fixed probability, it suffices to show that i gets a with (weakly) higher probability when he ranks a first. Let $\mathbf{Q} \in \mathcal{R}^N$. If $i \in N_{\mathbf{Q},a}$, then we show that the probability of i getting a weakly decreases if he changes his preference. In the profile $\mathbf{Q}$, the probability that i gets a is $\varphi(N_{\mathbf{Q},a})$, and if he changes his preference, his probability of getting a becomes $g_\varphi(N_{\mathbf{Q},a} \setminus \{i\})$, which is weakly lower by the hypothesis, as required. Similarly, if $i \notin N_{\mathbf{Q},a}$ then the probability of i getting a is $g_\varphi(N_{\mathbf{Q},a})$, and if he changes his preference, it becomes $\varphi(N_{\mathbf{Q},a} \cup \{i\})$, which is weakly higher, as required. Thus, under the aforementioned constraint on φ, $f(\varphi)$ satisfies SP, as desired.

For the second step, we will show that every $\varphi \in \mathcal{F}$ satisfies this constraint. Indeed, let $R \subseteq N$ be nonempty and let $i \in R$. Since $\varphi(R) \geq \frac{1}{n}$ for any nonempty $R \subseteq N$, it suffices to show that

$g_\varphi(R \setminus \{i\}) \leq \frac{1}{n}$. If $R = \{i\}$ then $g_\varphi(R \setminus \{i\}) = g_\varphi(\varnothing) = \frac{1}{n}$. Otherwise we have:

$$g_\varphi(R \setminus \{i\}) = \frac{1 - |R \setminus \{i\}| \, \varphi(R \setminus \{i\})}{n - |R \setminus \{i\}|} \leq \frac{1 - |R \setminus \{i\}| \frac{1}{n}}{n - |R \setminus \{i\}|} = \frac{1}{n}$$

Thus, the constraint is satisfied. In conclusion, for every $\varphi \in \mathcal{F}$, $f(\varphi)$ satisfies SP.

Regarding ExPE, we will show that for every $\varphi \in \mathcal{F}$, $f(\varphi)$ coincides with f_M for some extensive-form mechanism M that can be expressed as a convex combination of the efficient mechanisms $\{\mathrm{SD}_\sigma\}_{\sigma \in S_n}$. First, note that since there are only two houses, the only aspect of the order of the agents that matters is the identity of the first two agents to arrive. Considering this, for $h_1, h_2 \in H$, we define:

$$\Sigma^{\mathbf{Q}}_{h_1,h_2} \coloneqq \{\sigma \in S_n \mid \forall i \in [2] : \sigma(i) \in N_{\mathbf{Q},h_i}\}.$$

Using this notation, we define M as follows:

$$M(\mathbf{Q}) \coloneqq \sum_{\sigma \in S_n} w^{\mathbf{Q}}_{\sigma,\varphi} \mathrm{SD}_\sigma(\mathbf{Q}),$$

where:

$$w^{\mathbf{Q}}_{\sigma,\varphi} \coloneqq \begin{cases} \frac{\frac{2}{n} - \varphi\left(N_{\mathbf{Q},a}\right)}{\left(n_{\mathbf{Q},a} - 1\right)(n-2)!} & \sigma \in \Sigma^{\mathbf{Q}}_{a,a} \\ \frac{g_\varphi\left(N_{\mathbf{Q},a}\right)}{\left(n_{\mathbf{Q},b} - 1\right)(n-2)!} & \sigma \in \Sigma^{\mathbf{Q}}_{b,b} \\ \frac{2}{n_{\mathbf{Q},b}(n-2)!} \left(\varphi\left(N_{\mathbf{Q},a}\right) - \frac{1}{n}\right) & \sigma \in \Sigma^{\mathbf{Q}}_{a,b} \\ 0 & \text{otherwise} \end{cases}$$

First, we will show that the weights are well-defined and non-negative. If $\Sigma^{\mathbf{Q}}_{a,a} \neq \varnothing$, then $n_{\mathbf{Q},a} \geq 2$, so we have $\frac{2}{n} - \varphi(N_{\mathbf{Q},a}) \geq 0$ and $n_{\mathbf{Q},a} - 1 \geq 1$. Similarly, if $\Sigma^{\mathbf{Q}}_{b,b} \neq \varnothing$ we have that $n_{\mathbf{Q},b} - 1 \geq 1$ and $g_\varphi(N_{\mathbf{Q},a}) = \frac{1 - n_{\mathbf{Q},a}\varphi\left(N_{\mathbf{Q},a}\right)}{n_{\mathbf{Q},b}}$, which is non-negative. If $\Sigma^{\mathbf{Q}}_{a,b} \neq \varnothing$ then $N_{\mathbf{Q},a} \notin \{\varnothing, N\}$, so $n_{\mathbf{Q},b} \geq 1$ and $\varphi(N_{\mathbf{Q},a}) \geq \frac{1}{n}$. Next, we will show that the sum of the weights equals 1. We note that $\left|\Sigma^{\mathbf{Q}}_{a,a}\right| = n_{\mathbf{Q},a}(n_{\mathbf{Q},a} - 1)(n-2)!$, $\left|\Sigma^{\mathbf{Q}}_{b,b}\right| = n_{\mathbf{Q},b}(n_{\mathbf{Q},b} - 1)(n-2)!$ and

$\left|\Sigma_{a,b}^{\mathbf{Q}}\right| = n_{\mathbf{Q},a} n_{\mathbf{Q},b} (n-2)!$. Therefore, for every $\mathbf{Q} \in \mathcal{R}^N$, the sum of the weights is given by:

$$\begin{aligned}\sum_{\sigma \in S_n} w_{\sigma,\varphi}^{\mathbf{Q}} &= \sum_{\sigma \in \Sigma_{a,a}^{\mathbf{Q}}} w_{\sigma,\varphi}^{\mathbf{Q}} + \sum_{\sigma \in \Sigma_{b,b}^{\mathbf{Q}}} w_{\sigma,\varphi}^{\mathbf{Q}} + \sum_{\sigma \in \Sigma_{a,b}^{\mathbf{Q}}} w_{\sigma,\varphi}^{\mathbf{Q}} \\ &= n_{Q,a} (n_{Q,a} - 1) (n-2)! \frac{\frac{2}{n} - \varphi (N_{Q,a})}{(n_{Q,a} - 1)(n-2)!} \\ &\quad + n_{Q,b} (n_{Q,b} - 1)(n-2)! \frac{g_\varphi (N_{Q,a})}{(n_{Q,b} - 1)(n-2)!} \\ &\quad + n_{Q,a} n_{Q,b} (n-2)! \cdot \frac{2}{n_{Q,b} (n-2)!} \left(\varphi (N_{Q,a}) - \frac{1}{n} \right) \\ &= \frac{2 n_{\mathbf{Q},a}}{n} - n_{\mathbf{Q},a} \varphi (N_{\mathbf{Q},a}) + n_{\mathbf{Q},b} \cdot \frac{1 - n_{\mathbf{Q},a} \varphi (N_{\mathbf{Q},a})}{n - n_{\mathbf{Q},a}} + 2 n_{\mathbf{Q},a} \varphi (N_{\mathbf{Q},a}) - \frac{2 n_{\mathbf{Q},a}}{n} \\ &= 1\end{aligned}$$

Now, since M is a convex combination of $\{\mathrm{SD}_\sigma\}_{\sigma \in S_n}$, f_M satisfies ExPE and SP (as a convex combination of mechanisms satisfying ExPE and SP).

Furthermore, ETE holds because of the symmetry in how the weights are defined. Therefore, by using the fact that each agent is assigned a house with probability $\frac{2}{n}$, along with ETE and complementing to 1 , in order to prove that $f(\varphi) = f_M$, it suffices to prove that f_M assigns the same probability as $f(\varphi)$ for an agent in $N_{\mathbf{Q},a}$ to receive house a , i.e., the probability $\varphi(N_{\mathbf{Q},a})$.

Indeed, for any $\mathbf{Q} \in \mathcal{R}^N$, an agent $i \in N_{\mathbf{Q},a}$ gets a in the following cases: when the order is in $\Sigma_{a,a}^{\mathbf{Q}}$ and agent i is first, with $n_{\mathbf{Q},a} - 1$ options for the second agent from $N_{\mathbf{Q},a} \setminus \{i\}$ and $(n-2)!$ options for the arrangement of the other agents. Additionally, agent i also gets the house a when the order is in $\Sigma_{a,b}^{\mathbf{Q}}$ and agent i is selected from $N_{\mathbf{Q},a}$, with $n_{\mathbf{Q},b}$ possible choices for the second agent from $N_{\mathbf{Q},b}$, and $(n-2)!$ options for the arrangement of the other agents. Thus, the probability that the agent gets a is:

$$(n_{\mathbf{Q},a} - 1)(n-2)! \cdot \frac{\frac{2}{n} - \varphi (N_{\mathbf{Q},a})}{(n_{\mathbf{Q},a} - 1)(n-2)!} + n_{\mathbf{Q},b} (n-2)! \cdot \frac{2}{n_{\mathbf{Q},b} (n-2)!} \left(\varphi (N_{\mathbf{Q},a}) - \frac{1}{n} \right) = \varphi (N_{\mathbf{Q},a})$$

Therefore, the equality $f(\varphi) = f_M$ holds, and $f(\varphi)$ satisfies ExPE as well. ■

Remark 5. For $n = 2, 3$ there is a unique function $\varphi \in \mathcal{F}$, hence a unique mechanism satisfying the axioms. However, when $n \geq 4$, for every subset $R \subseteq N$ of agents of size $2 \leq |R| \leq n-2$, we obtain an interval of positive length for the options for $\varphi(R)$. Therefore, the dimension of $\mathcal{F}$ is given by $\sum_{i=2}^{n-2} \binom{n}{i} = 2^n - 2n - 2$. Moreover, if we were to strengthen the axiom of ETE to anonymity, the dimension would decrease to $\sum_{i=2}^{n-2} 1 = n - 3$, as there would be only one degree of freedom for each subset size. If we also required neutrality, the dimension would drop further to $\left\lceil \frac{n-3}{2} \right\rceil$, since the value of φ on sets larger than $\frac{n}{2}$ would be determined by its value on their complements.

Remark 6. In the proof of Proposition 3.1, the first part relies only on lower invariance, whereas the second part establishes SP. Consequently, Proposition 3.1 remains valid if SP is replaced by lower invariance.

Remark 7. The function corresponding to the RSD mechanism is

$$\varphi_{\mathrm{RSD}}(R) = \frac{1}{n} + \frac{n-|R|}{(n-1)\,n}.$$

The first term, $\frac{1}{n}$, accounts for the permutations in which the agent appears first and thus selects their top choice. The second term accounts for the permutations where the agent appears second, following an agent who prefers the house b, thereby still allowing the agent to choose house a.

Moreover, we observe that when $n \geq 4$, RSD is dominated by the mechanism corresponding to the function

$$\varphi(R) = \min\left\{\frac{2}{n}, \frac{1}{|R|}\right\},$$

which assigns to every agent their preferred house with the maximum probability allowed under the axioms.

Remark 8. The domination phenomenon observed above for the case $m = 2$ with $n \geq 4$ extends to markets with $m \geq 2$ and $n \geq m + 2$ via an extension-and-symmetrization argument. Fix such a pair (n, m), and set $\alpha := m - 2$. Starting from a mechanism f on the $(n - \alpha, 2)$ market that dominates f_{RSD} (for example, the mechanism corresponding to the function φ established in the remark above), we extend it to the (n, m) market by adding α auxiliary agents and α new houses. The auxiliary agents are ordered first, and each selects their most preferred remaining house. After these α steps, if the set of remaining houses coincides exactly with the original set of 2 houses (equivalently, the auxiliary agents have selected exactly the α added houses), we apply f to the $n-\alpha$ original agents with their preferences restricted to these 2 houses; otherwise, we apply f_{RSD} to the $n - \alpha$ original agents with preferences restricted to the remaining houses. Finally, we symmetrize the resulting rule by uniformly averaging over all renamings of the agents, which restores anonymity (and hence ETE) while preserving the other axioms. This construction yields a mechanism satisfying the axioms that dominates f_{RSD} in the (n, m) market. We omit the details.

4 Cases where there is uniqueness

This section identifies the values of m and n for which the axioms ExPE, ETE, and SP jointly determine a unique normal-form mechanism. That is, for such values of m and n, the axioms are sufficiently strong to uniquely determine the assignment matrix for every possible preference profile.

Remark 9. Throughout this section, ExPE is used only via the weaker conditions of support efficiency (Definition 2.8) and the full assignment property (Definition 2.9), and SP is used only via upper and lower invariance (Definition 2.10). The reader may verify that the arguments rely only on these weaker conditions.

Remark 10. When $n = 2$ and $m > 2$, the axioms uniquely determine the mechanism. Indeed, ExPE implies that each agent receives one of his top two choices with probability 1, so it suffices to determine the probability that each agent receives his top choice. If the two agents share the same top choice, then ETA implies that each receives their common top choice with probability

$\frac{1}{2}$. Otherwise, the unique efficient deterministic assignment assigns each agent his top choice, and hence both agents receive their respective top choices with probability 1 .

4.1 $n = 3$ and $m > 3$

We focus here on the case $n = 3$ and $m > 3$. In the balanced case $n = m = 3$, the result was already established by Bogomolnaia and Moulin [8]; nevertheless, the proof below applies verbatim to $m = 3$ and thus also yields an alternative proof of their result.

Proposition 4.1. For $n = 3$ and $m \geq 3$, the axioms ExPE, ETE, and SP uniquely characterize f_{RSD} .

Proof. Let f be a mechanism satisfying the axioms. We will prove that f necessarily coincides with f_{RSD} . To this end, we focus on profiles where f differs from f_{RSD} .

First, we claim that if there exists a profile where f differs from f_{RSD} , then there also exists such a profile in which all agents rank the same house as their first choice. Second, we show that f cannot differ from f_{RSD} at such a profile.

For the first part, let $\mathbf{P}$ be a profile such that $f(\mathbf{P}) \neq f_{\text{RSD}}(\mathbf{P})$. If all three agents rank the same house first, we may proceed directly to the second part. Otherwise, note that by efficiency, if the three agents rank three different houses first, then the probabilities are uniquely determined at this profile. Thus, it must be the case that two agents rank the same house first, and the third agent ranks a different house first. Without loss of generality, suppose that agents 1 and 2 rank house a first, while agent 3 ranks house b first. Since $f(\mathbf{P}) \neq f_{\text{RSD}}(\mathbf{P})$, there must exist some agent for whom the two mechanisms assign different distributions.

Suppose that $f(\mathbf{P})_3 \neq f_{\text{RSD}}(\mathbf{P})_3$. Let x be a house for which the two mechanisms differ, that is, $f(\mathbf{P})_{x,3} \neq f_{\text{RSD}}(\mathbf{P})_{x,3}$. Note that $x \neq a$, because by efficiency, agent 3 is not assigned house a . Now, consider a sequence of profiles starting from $\mathbf{P}$, where at each step we perform a swap in agent 3 's preferences between house a and the house ranked immediately above it, until we reach a profile $\mathbf{Q}$ in which agent 3 ranks a first. We claim that $f(\mathbf{Q})_{x,3} \neq f_{\text{RSD}}(\mathbf{Q})_{x,3}$ as well.

To see this, observe that the axioms determine the probability that agent 3 is assigned house a at each profile along the sequence: in the final profile $\mathbf{Q}$, agent 3 is assigned a with probability $\frac{1}{3}$ by ETA and efficiency (since efficiency ensures that a is assigned with probability 1). In all earlier profiles, efficiency guarantees that agent 3 is not assigned a . Thus, the additional probability that agent 3 gains for house a after each swap is fully determined by the axioms, and by SP it corresponds exactly to the reduction in the probability of the house with which a was swapped. In particular, for the swap involving a and x (if such a swap occurred), the change in x 's probability is determined, while by SP, all other swaps do not affect the probability of x . Therefore, we conclude that $f(\mathbf{Q})_{x,3} - f_{\text{RSD}}(\mathbf{Q})_{x,3} = f(\mathbf{P})_{x,3} - f_{\text{RSD}}(\mathbf{P})_{x,3}$ and hence $f(\mathbf{Q})_{x,3} \neq f_{\text{RSD}}(\mathbf{Q})_{x,3}$, as claimed.

Otherwise, $f(\mathbf{P})_3 = f_{\text{RSD}}(\mathbf{P})_3$ and the difference between the mechanisms lies in the distribution of another agent. Without loss of generality, suppose it is agent 1 . By ETA, the probability that agent 1 is assigned house a is $\frac{1}{2}$. Furthermore, by efficiency, agent 1 cannot be assigned houses that are not among their top 3 preferences. Since agent 1 is assigned some house with probability 1 (because there are more houses than agents), there must be at least two houses with different probabilities, and these houses must be the second and third preferences in agent 1's ranking.

Note that the probability of the third-ranked house is not determined by the axioms, so it must be efficient for agent 1 to receive this house. In other words, there must be an ordering of the agents where agent 1 takes their third choice. However, agent 1 can only take their third choice if they are last in the ordering. In this case, agents 2 and 3 must be assigned houses a and b, respectively. For agent 1 to receive their third choice, agent 1's top two preferences must be a and b. Hence, agent 1 must rank b second.

At this point, by the earlier arguments, we have $f(\mathbf{P})_{b,1} \neq f_{\text{RSD}}(\mathbf{P})_{b,1}$, and since agent 3 ranks b first, house b must be assigned to some agent with probability 1. Thus, there must be some other agent where the allocation probability of b differs between the two mechanisms. Since $f(\mathbf{P})_3 = f_{\text{RSD}}(\mathbf{P})_3$, this agent must be agent 2, and the difference in probabilities must cancel the difference observed for agent 1.

By applying the same analysis to agent 2, we conclude that agent 2 must rank b second. Now, since agents 1 and 2 rank a first and b second, and agent 3 ranks b first, we have $f_{\text{RSD}}(\mathbf{P})_{b,i} = \frac{1}{6}$ for $i = 1, 2$. Moreover, since we have shown that the differences for the two agents must cancel, it follows that $f(\mathbf{P})_{b,2} \neq f_{\text{RSD}}(\mathbf{P})_{b,2}$ and $f(\mathbf{P})_{b,2} \neq f(\mathbf{P})_{b,1}$. Hence, by ETE, their preferences must differ.

However, since both agents rank a first and b second, we can make their preferences similar by applying a sequence of swaps to agent 2's preference order that do not involve b. Let $\mathbf{P}'$ denote the profile obtained after these swaps. Since these swaps do not involve b, by SP, the difference in probabilities remains unchanged, i.e., $f(\mathbf{P})_{b,2} - f_{\text{RSD}}(\mathbf{P})_{b,2} = f(\mathbf{P}')_{b,2} - f_{\text{RSD}}(\mathbf{P}')_{b,2}$. Moreover, since in $\mathbf{P}'$, agents 1 and 2 have identical preferences, their differences must also be the same, i.e., $f(\mathbf{P}')_{b,2} - f_{\text{RSD}}(\mathbf{P}')_{b,2} = f(\mathbf{P}')_{b,1} - f_{\text{RSD}}(\mathbf{P}')_{b,1}$. Since the total probability of b is 1 in $\mathbf{P}'$, in order to cancel the differences, we must have $f(\mathbf{P}')_{b,3} \neq f_{\text{RSD}}(\mathbf{P}')_{b,3}$. Thus, $f(\mathbf{P}')_3 \neq f_{\text{RSD}}(\mathbf{P}')_3$, and we can apply the same arguments of the first case (where $f(\mathbf{P})_3 \neq f_{\text{RSD}}(\mathbf{P})_3$) to obtain a profile $\mathbf{Q}'$ where all agents rank the same house first and $f(\mathbf{Q}') \neq f_{\text{RSD}}(\mathbf{Q}')$, as claimed.

For the second part, let $\mathbf{R}$ be a profile in which all agents rank the same house first, and suppose $f(\mathbf{R}) \neq f_{\text{RSD}}(\mathbf{R})$. Without loss of generality, assume that this top-ranked house is a, and that the allocation differs at agent 1. As in the previous part, the difference must occur in the probabilities assigned to 1's second and third choices, and thus, their third-ranked house must be efficient for them. Let b denote agent 1's second choice. Then, for agent 1's third choice to be efficient, b must be taken before agent 1 chooses, which requires at least one other agent to rank b second. Without loss of generality, suppose this is agent 2.

Now, regardless of whether agent 3 ranks b second or not, the allocation probabilities for house b must be fully determined by ETA (if agent 3 does not rank b second, then by efficiency, he cannot receive it). But this contradicts the assumption that agent 1's allocation differs from f_{RSD} at house b. Therefore, the assumption that such a profile $\mathbf{R}$ exists must be false. It follows, by the arguments above, that $f = f_{\text{RSD}}$, as desired. ∎

4.2 $n = m = 4$

We focus here on the balanced case $n = m = 4$. The result is not new: it has been confirmed via computer-aided proofs (see, e.g., Sandomirskiy [17]). The proof below is purely analytic, but it requires a detailed case analysis.

Proposition 4.2. For $n = m = 4$, the axioms ExPE, ETE, and SP uniquely characterize f_{RSD}.

Proof. We prove the proposition by showing that for each profile $\mathbf{P} \in \mathcal{R}^N$, the assignment matrix is uniquely determined by the axioms ExPE, ETE, and SP. The proof proceeds by induction on

$$d = d_{\mathbf{P}} := \left| \left\{ (\{i, j\}, \{h_1, h_2\}) \in \binom{N}{2} \times \binom{H}{2} \mid \begin{array}{c} i \text{ and } j \text{ disagree on the binary} \\ \text{preference between } h_1 \text{ and } h_2 \end{array} \right\} \right|.$$

Intuitively, $d_{\mathbf{P}}$ quantifies the extent of disagreement present in the profile $\mathbf{P}$.

In the base case $d = 0$, all agents have identical rankings over the houses. In this case, ExPE and ETE directly determine the assignment matrix.

Now suppose $\mathbf{P} \in \mathcal{R}^N$ is a profile with $d = d_{\mathbf{P}} > 0$. We first observe that the assignment of agent i in $\mathbf{P}$ is determined under the induction hypothesis if the following condition $(*)_{\mathbf{P},i}$ holds: there exists at least one pair $\{h_1, h_2\}$ of adjacent houses in P_i such that less than two other agents agree with agent i on the binary preference between those houses; and furthermore, if there is exactly one such pair, then at least one of the houses in that pair cannot be assigned to agent i in any efficient assignment with respect to $\mathbf{P}$.

To show this, suppose we find such a pair $\{h_1, h_2\}$ of adjacent houses in agent i's ranking. Consider the profile $\mathbf{P}'$ obtained by swapping the positions of h_1 and h_2 in i's ranking. This operation reduces the number of disagreements in the profile, so $d_{\mathbf{P}'} < d$. By the induction hypothesis, the assignment matrix for $\mathbf{P}'$ is uniquely determined by the axioms. By SP, the probability that agent i receives any house other than h_1 or h_2 must remain unchanged between $\mathbf{P}$ and $\mathbf{P}'$, and is therefore determined in $\mathbf{P}$ as well. Since $n = m$, agent i must be assigned to some house. Therefore, once the probabilities that agent i receives the two houses outside the pair $\{h_1, h_2\}$ are known, it suffices to determine either the probability that i receives h_1 or the probability that he receives h_2.

If there exists another such pair of adjacent houses in agent i's ranking, then without loss of generality, we may assume that this second pair does not involve h_1. Applying the same reasoning as before, we can determine the probability that agent i receives h_1, which completes the determination of his assignment. If no such additional pair exists, then by assumption, one of the houses in the original pair cannot be assigned to agent i under any efficient assignment. In that case, the probability that agent i receives that house is zero, and thus his assignment is fully determined. We have therefore shown that under condition $(*)_{\mathbf{P},i}$, the assignment of agent i in profile $\mathbf{P}$ is uniquely determined by the axioms.

Since $n = m$, every house must be assigned with probability 1. Therefore, to determine the entire assignment matrix of $\mathbf{P}$, it suffices to determine the assignments of $n-1 = 3$ agents. More generally, it suffices to determine the assignments of all agents except a set of agents that have identical rankings, by ETE. By the argument above, the assignment of any agent satisfying the condition is fully determined.

Hence, it remains to consider only those profiles $\mathbf{P}$ in which there exist at least two agents $i \in N$ with different rankings who do not satisfy $(*)_{\mathbf{P},i}$. For clarity, we introduce the following definitions.

Definition 4.1 (Supported agent)**.** Fix a profile $\mathbf{P}$. An agent $i \in N$ is *supported* (with respect to $\mathbf{P}$) if for every pair $\{h_1, h_2\}$ of adjacent houses in P_i, there are at least two agents other than i who agree with agent i on the binary preference between $\{h_1, h_2\}$, possibly with a single exception: there may be one pair of adjacent houses $\{h_1, h_2\}$ for which fewer than two other agents agree with i, provided that both h_1 and h_2 are assigned to agent i in some efficient assignment with respect to $\mathbf{P}$.

Remark 11. Note that, if such an exceptional pair exists, then there must be exactly one other agent who agrees with agent i on the binary preference between h_1 and h_2. Otherwise, by efficiency, agent i could not be assigned the house he ranks lower in that pair, contradicting the assumption that both houses must be assigned to him in some efficient assignment.

Definition 4.2 (Supported profile)**.** A profile $\mathbf{P}$ is *supported* if there exist at least two supported agents with respect to $\mathbf{P}$ whose rankings are different.

Thus, in the terminology above, the class of profiles that remain under consideration are precisely the supported profiles. We complete the proof by exhaustively verifying that in all such profiles, the assignment matrix is uniquely determined by the axioms. The exhaustive case analysis appears in Appendix B. ■

5 Cases where there are other mechanisms

To better understand the limitations of the axioms ExPE, ETE, and SP, this section identifies the values of m and n for which these axioms do *not* suffice to characterize f_{RSD}. That is, for such values of m and n, the axioms are too weak to uniquely determine the assignment matrix for every possible preference profile. The case $m = 2$ was already resolved earlier, and the case $n, m \geq 5$ has been shown by Basteck and Ehlers [5]. The mechanisms we construct in this section rely on ideas similar to those used in their work, and they cover the remaining settings, namely $n > m \in \{3, 4\}$ and $n = 4$, $m \geq 5$.

5.1 $n > m \in \{3, 4\}$

Proposition 5.1. For $n > m \geq 3$, the axioms ExPE, ETE, and SP do not suffice to characterize the mechanism f_{RSD}.

Proof. To establish the proposition, we construct a mechanism that satisfies the axioms but differs from f_{RSD}. Let $H := \{h_1, \ldots, h_m\}$. We begin with some preliminary notations before specifying the mechanism.

We denote by f^1 the mechanism corresponding to choosing uniformly an ordering of the agents in which agent 1 comes first. Formally, f^1 is the normal form of the extensive-form mechanism

$$M^1 \coloneqq \frac{1}{(n-1)!} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \mathrm{SD}_\sigma .$$

Note that RSD can be obtained by symmetrizing M^1 over all renamings of the agents, and therefore

$$f_{\mathrm{RSD}} = \frac{1}{n!} \sum_{\pi \in \Pi} \pi\left(f^1\right).$$

For a preference profile $\mathbf{P} \in \mathcal{R}^N$ and an agent $i \in N$, let $\operatorname{top}(P_i)$ denote the house most preferred by agent i in $\mathbf{P}$. Let $x = x_{\mathbf{P}}$ denote the second-best house of agent 1 in $\mathbf{P}$, and set $a \coloneqq h_1$. We then define

$$\mathsf{P} \coloneqq \left\{ \mathbf{P} \in \mathcal{R}^N \mid \begin{array}{c} \forall i \in [m] : \operatorname{top}(P_i) = h_i, \\ a P_n x \end{array} \right\}.$$

Note that $x \neq a$ for every $\mathbf{P} \in \mathsf{P}$.

Let $\varepsilon > 0$ be sufficiently small (e.g., $\varepsilon < \frac{1}{(n-1)!}$ suffices), and define the vector $v = (v_h)_{h \in H}$ by

$$v_h \coloneqq \begin{cases} \varepsilon & \text{if } h = x, \\ -\varepsilon & \text{if } h = a, \\ 0 & \text{otherwise.} \end{cases}$$

Now define the mechanism $f^{1,n}$ by

$$f^{1,n}(\mathbf{P})_i \coloneqq \begin{cases} f^1(\mathbf{P})_i + v & \text{if } i = 1 \text{ and } \mathbf{P} \in \mathsf{P}, \\ f^1(\mathbf{P})_i - v & \text{if } i = n \text{ and } \mathbf{P} \in \mathsf{P}, \\ f^1(\mathbf{P})_i & \text{otherwise.} \end{cases}$$

The mechanism is well defined: the row sums and column sums remain unchanged by construction, and the entries remain nonnegative. To see the latter, note that in f^1, for profiles in P, agent 1 receives a with probability one, while agent n receives x in at least one ordering where agent 1 comes first. Specifically, if $x = h_i$, then agent n receives x in the ordering where the agents in $[m] \setminus \{i\}$ appear first in their natural order, and agent n comes immediately after them. Hence, in f^1 agent n obtains x with probability at least $\frac{1}{(n-1)!}$. Thus, the adjustment by v preserves feasibility while slightly shifting the allocation between agents 1 and n.

With these notations, define

$$f \coloneqq \frac{1}{n!} \sum_{\pi \in \Pi} \pi\left(f^{1,n}\right).$$

We claim that f is the desired mechanism, that is, it satisfies the axioms ExPE, ETE, and SP, yet it differs from f_{RSD}. First, by construction, since it is defined by symmetrizing over all agent renamings, it clearly satisfies ETE (and even anonymity). For ExPE and SP, it suffices to verify that $f^{1,n}$ satisfies them, because these properties are invariant under renamings and preserved under convex combinations.

For ExPE, it is clear that $f^{1,n}$ satisfies the axiom whenever $\mathbf{P} \notin \mathsf{P}$, since in that case $f^{1,n}(\mathbf{P}) = f^{1}(\mathbf{P})$ and f^{1} satisfies ExPE by construction. We therefore focus on the case $\mathbf{P} \in \mathsf{P}$. Let $x \coloneqq h_i$ with $i \neq 1$. Consider the two assignments $s, s' : N \to O$ defined by

$$s(j) \coloneqq \begin{cases} h_j & \text{if } j \in [m] \setminus \{i\}, \\ x & \text{if } j = n, \\ \varnothing & \text{otherwise,} \end{cases} \qquad s'(j) \coloneqq \begin{cases} x & \text{if } j = 1, \\ a & \text{if } j = n, \\ s(j) & \text{otherwise.} \end{cases}$$

Here s' differs from s only by swapping the houses a and x between agents 1 and n. Transferring ε weight from s to s' describes exactly the adjustment needed to pass from $f^{1}(\mathbf{P})$ to $f^{1,n}(\mathbf{P})$.

Since f^{1} satisfies ExPE, to show that $f^{1,n}$ satisfies ExPE it remains to verify that this transfer is between efficient assignments, and that the transferred weight ε is less than the weight placed on s by some ex-post efficient extensive-form mechanism whose normal form is f^{1}. Because ε is chosen sufficiently small and M^{1} is such an extensive-form mechanism, it suffices to show that s can be obtained from an ordering of the agents where agent 1 comes first, and that s' can be obtained from some ordering.

First, the assignment s can be obtained by an ordering in which the agents in $[m] \setminus \{i\}$ appear first in their natural order, and agent n comes immediately after them. This is indeed an ordering where agent 1 comes first.

Second, let $I \subseteq [m]$ be the set of indices of the houses that agent n strictly prefers to a. Note that $1 \notin I$ since $h_1 = a$, and $i \notin I$ since $\mathbf{P} \in \mathsf{P}$ implies that agent n prefers a over $x = h_i$. Then s' can be obtained by an ordering where the agents in I come first (in an arbitrary order), followed by agent n, then agent 1, and then the remaining agents in $[m] \setminus \{i\}$, all before any of the other agents. Therefore, $f^{1,n}$ satisfies ExPE.

The next step is to verify that $f^{1,n}$ also satisfies SP. Since f^{1} satisfies SP, agents $2, \dots, n-1$ cannot gain from any unilateral misreport. We therefore need only check agents 1 and n. By the local characterization of SP, it suffices to consider misreports in the form of a swap between two adjacent houses. Let $\mathbf{Q}$ be the profile before the swap and $\mathbf{R}$ the profile after it.

Since f^{1} already satisfies SP, the only potentially problematic cases are those where

$$f^{1,n}(\mathbf{Q}) - f^{1}(\mathbf{Q}) \neq f^{1,n}(\mathbf{R}) - f^{1}(\mathbf{R}),$$

so that the modification in $f^{1,n}$ could change incentives, and it suffices to verify that, taking $\mathbf{Q}$ as the profile of true preferences, the difference under $\mathbf{Q}$ is at least as favorable for the swapping agent as the difference under $\mathbf{R}$. We will refer to this difference as the adjustment, meaning the ε weight transfer that specifies how $f^{1,n}$ deviates from f^{1} at a given profile.

Now, because $f^{1}(\mathbf{P}) = f^{1,n}(\mathbf{P})$ for every $\mathbf{P} \notin \mathsf{P}$, we must have at least one of $\mathbf{Q}, \mathbf{R}$ in P. Moreover, the roles of $\mathbf{Q}$ and $\mathbf{R}$ can be interchanged: under the local characterization of SP, the condition is expressed as a collection of equalities together with a single inequality, and that inequality simply reverses when the two houses are swapped. Hence, without loss of generality, we may assume $\mathbf{Q} \in \mathsf{P}$. With these reductions, it remains to check only those adjacent swaps by agents 1 and n that actually affect the adjustment.

For agent 1, since $\mathbf{Q} \in \mathsf{P}$, his first and second choices are a and x, respectively. Any swap not involving one of these houses leaves the adjustment unchanged. Let y denote his third-preferred house in $\mathbf{Q}$. It remains to consider two cases:

Swap between a and x: here we directly compare $f^{1,n}(\mathbf{Q})$ and $f^{1,n}(\mathbf{R})$. In $\mathbf{Q}$, agent 1 receives a with probability $1-\varepsilon$ and x with probability ε. In $\mathbf{R}$, he receives x with probability 1, which is strictly worse for him. Note also that the weight transfer occurs exactly between the two houses involved in the swap.

Swap between x and y: here we compare the adjustments mentioned earlier. In $\mathbf{Q}$, the adjustment from f^1 to $f^{1,n}$ transfers ε weight from a to x. In $\mathbf{R}$, the adjustment transfers ε weight from a to y. Since agent 1 prefers x over y, this is less favorable for him. Moreover, the difference between the adjustments involves only the houses in the swap - it consists of an ε weight transfer from x to y.

Thus, agent 1 cannot gain from unilateral manipulation. For agent n, the only swap that can matter is between a and x. In this case we move from a profile $\mathbf{Q} \in \mathsf{P}$ where the adjustment transfers ε weight from x to a (which he prefers), to a profile $\mathbf{R} \notin \mathsf{P}$, where no transfer occurs. This is again less favorable, and the difference concerns only a and x, the houses in the swap. Therefore, agent n cannot gain from a unilateral manipulation either. Hence, f satisfies ExPE, ETE, and SP.

Finally, it remains to observe that $f \neq f_{\text{RSD}}$. Fix a profile $\mathbf{Q} \in \mathsf{P}$ such that $x_{\mathbf{Q}} = h_2$ and no agent other than agent 1 ranks a as his top choice. Note that such a profile exists, by our assumption that $m \geq 3$. By the definition of $f^{1,n}$ on profiles in P, we have

$$f^{1,n}(\mathbf{Q})_{h_2,1} > f^1(\mathbf{Q})_{h_2,1}.$$

We now claim that for every $\pi \in \Pi$,

$$\pi\left(f^{1,n}\right)(\mathbf{Q})_{h_2,1} \geq \pi\left(f^1\right)(\mathbf{Q})_{h_2,1}.$$

Suppose otherwise, in which case for some π the adjustment from $\pi\left(f^1\right)$ to $\pi\left(f^{1,n}\right)$ would reduce the probability that agent 1 receives h_2. Since such a reduction can only occur when $\pi(\mathbf{Q}) \in \mathsf{P}$, this enforces agent $\pi^{-1}(1)$ to rank a as his first choice in $\mathbf{Q}$, so by the construction of $\mathbf{Q}$ we must have $\pi^{-1}(1) = 1$. However, the adjustment never reduces the probability that agent 1 receives his second-best house, a contradiction.

Hence, $\pi\left(f^{1,n}\right)(\mathbf{Q})_{h_2,1} \geq \pi\left(f^1\right)(\mathbf{Q})_{h_2,1}$ for all $\pi \in \Pi$, and strict inequality holds for $\pi = \text{id}$. Since f and f_{RSD} are the averages of $\left\{\pi\left(f^{1,n}\right)\right\}_{\pi\in\Pi}$ and $\left\{\pi\left(f^1\right)\right\}_{\pi\in\Pi}$, respectively, it follows that

$$f(\mathbf{Q})_{h_2,1} > f_{\text{RSD}}(\mathbf{Q})_{h_2,1}.$$

Therefore, $f \neq f_{\text{RSD}}$, so the axioms do not suffice to characterize f_{RSD} in such settings. ■

5.2 $n = 4$, $m \geq 5$

In the setting $n = 4$, $m \geq 5$, we show that f_{RSD} is not unique by presenting another mechanism satisfying the axioms. The idea is borrowed from [5]: we take the mechanism introduced there for five agents and restrict it to four agents by removing agent 5 and its allocation. We now formalize this construction in the following proposition.

Proposition 5.2. For $n = 4$, $m \geq 5$, the axioms ExPE, ETE, and SP do not suffice to characterize the mechanism f_{RSD}.

Proof. Let a, b, c, d, e denote five distinct houses. We extend the notation from the previous proof: for a preference profile $\mathbf{P}$, an agent $i \in N$, and $j \in [m]$, let $\mathrm{top}_j(P_i)$ denote the j-th most-preferred house of agent i under $\mathbf{P}$. Using this notation, define

$$\mathsf{P} := \left\{ \mathbf{P} \in \mathcal{R}^N \;\middle|\; \begin{array}{c} \forall h \in H \setminus \{a, b, e\} : aP_1h \text{ and } bP_1h, \\ \forall i \in N \setminus \{1\} : \mathrm{top}_1(P_i) = e \text{ and } \mathrm{top}_3(P_i) = c, \\ \mathrm{top}_2(P_2) = a,\ \mathrm{top}_2(P_3) = b,\ \mathrm{top}_2(P_4) = d \end{array} \right\}.$$

That is, P is the set of profiles in which agent 1's top two houses in $H \setminus \{e\}$ are a and b (in some order), while agents $2, 3, 4$ all rank e first and c third, with their second-ranked houses fixed as a, b, d, respectively. Denote by $x = x_{\mathbf{P}}$ the most-preferred house of agent 1 in $H \setminus \{a, b, e\}$ under profile $\mathbf{P}$, and by $y = y_{\mathbf{P}}$ his less-preferred house among $\{a, b\}$.

Let $\varepsilon > 0$ be sufficiently small (for instance, $\varepsilon < \frac{1}{4!}$ suffices), and define the vector $v = (v_h)_{h \in H}$ by

$$v_h := \begin{cases} \varepsilon & \text{if } h = y, \\ -\varepsilon & \text{if } h = x, \\ 0 & \text{otherwise.} \end{cases}$$

Now define a mechanism g by

$$g(\mathbf{P})_i := \begin{cases} f_{\mathrm{RSD}}(\mathbf{P})_i + v & \text{if } i = 1 \text{ and } \mathbf{P} \in \mathsf{P}, \\ f_{\mathrm{RSD}}(\mathbf{P})_i & \text{otherwise.} \end{cases}$$

In words, g coincides with f_{RSD} except on profiles in P, where agent 1's assignment is adjusted by transferring ε probability from house x to house y. The mechanism is well defined, since under the ordering 4231 (agents listed from first to last), agent 1 receives house x. We now obtain the desired mechanism f by symmetrizing g over all renamings of the agents:

$$f := \frac{1}{4!} \sum_{\pi \in \Pi} \pi(g),$$

and we will show that f satisfies the axioms and differs from f_{RSD}. Since f is obtained by symmetrizing g over all renamings, it satisfies ETE (and in fact anonymity). For the remaining axioms, as in the previous proof, it suffices to verify them for g.

For ExPE, note first that f_{RSD} satisfies the axiom, so it suffices to consider profiles $\mathbf{P} \in \mathsf{P}$. Since every efficient assignment with respect to $\mathbf{P}$ has probability at least ε in $\mathrm{RSD}(\mathbf{P})$, it is enough to construct an adjustment that shifts probability between efficient assignments so that agent 1's allocation transfers ε probability from x to y, while the allocations of the other agents remain unchanged. We distinguish two symmetric cases, according to whether aP_1b or bP_1a. We present the case aP_1b; the other follows by the same reasoning with the roles of agents 2 and 3 interchanged. In the case aP_1b, we have $y = b$. Consider the following four assignments, where each agent is matched with the house listed beneath him:

$$\begin{pmatrix}1 & 2 & 3 & 4\\ x & a & b & e\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ a & c & e & d\end{pmatrix} \to \begin{pmatrix}1 & 2 & 3 & 4\\ b & a & e & d\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ a & c & b & e\end{pmatrix}.$$

Reducing ε probability from each assignment on the left and adding ε probability to each assignment on the right yields exactly the desired adjustment: agent 1's allocation shifts ε probability from x to b, while the allocations of all the other agents remain unchanged. It remains to check that these four assignments are efficient. Indeed, the two assignments on the left are generated by the orderings 4231 and 3124, and the two on the right by 3214 and 4123. Hence, all are efficient assignments, and thus g satisfies ExPE.

For SP, as in the previous proof, it suffices to consider the adjacent swaps of agent 1 that can affect the adjustment from f_{RSD} to g. Let $\mathbf{Q}$ denote the profile of true preferences and $\mathbf{R}$ the profile after the swap. Without loss of generality (by the same argument as in the previous proof), we may assume that $\mathbf{Q} \in \mathsf{P}$. Moreover, by symmetry, we may assume that aQ_1b, and denote by c', d' the two most-preferred houses of agent 1 in $H \setminus \{a, b, e\}$, ordered so that $c'Q_1d'$. The only adjacent swaps of agent 1 that can influence the adjustment are those between the pairs $(a, b), (b, c'), (c', d')$. By the definition of g, in each case the difference between the adjustments under $\mathbf{Q}$ and $\mathbf{R}$ consists solely of a shift in probability between the two houses in the swapped pair. When the swapped pair is (b, c'), there is no adjustment under $\mathbf{R}$, which is worse for agent 1 than the adjustment under $\mathbf{Q}$; hence he cannot gain from this deviation. For the swaps (a, b) and (c', d'), the adjustment under $\mathbf{R}$ is actually more favorable to agent 1, which presents the potential difficulty.

When the swapped pair is (c', d'), the change in the adjustment from $\mathbf{Q}$ to $\mathbf{R}$ shifts ε probability from d' to c'. Hence,

$$\left(\sum_{h\in C_{Q_1}(h')} g(\mathbf{R})_{1,h} - \sum_{h\in C_{Q_1}(h')} f_{\text{RSD}}(\mathbf{R})_{1,h}\right) - \left(\sum_{h\in C_{Q_1}(h')} g(\mathbf{Q})_{1,h} - \sum_{h\in C_{Q_1}(h')} f_{\text{RSD}}(\mathbf{Q})_{1,h}\right) = \begin{cases}\varepsilon & \text{if } h' = c',\\ 0 & \text{otherwise.}\end{cases}$$

Since f_{RSD} satisfies SP, for every $h' \neq c'$ we already have

$$\sum_{h\in C_{Q_1}(h')} g(\mathbf{R})_{1,h} \leq \sum_{h\in C_{Q_1}(h')} g(\mathbf{Q})_{1,h}.$$

To establish the same inequality for $h' = c'$, it suffices to show that

$$\sum_{h\in C_{Q_1}(c')} f_{\text{RSD}}(\mathbf{R})_{1,h} \leq \left(\sum_{h\in C_{Q_1}(c')} f_{\text{RSD}}(\mathbf{Q})_{1,h}\right) - \varepsilon.$$

By the definition of RSD and the choice of ε, this reduces to showing that

$$\left|\{\sigma \in S_n \mid \mathrm{SD}_\sigma(\mathbf{R})(1) \in C_{Q_1}(c')\}\right| < \left|\{\sigma \in S_n \mid \mathrm{SD}_\sigma(\mathbf{Q})(1) \in C_{Q_1}(c')\}\right|.$$

That is, the number of orderings in which agent 1 receives a house in $C_{Q_1}(c')$ strictly decreases when moving from $\mathbf{Q}$ to $\mathbf{R}$. Indeed, the set corresponding to $\mathbf{R}$ is contained in that for $\mathbf{Q}$, and they are not equal: for example, the ordering 4231 belongs to the set for $\mathbf{Q}$ but not to the one for $\mathbf{R}$. Hence, $g(\mathbf{R})_1 \preceq_{Q_1} g(\mathbf{Q})_1$. For the swapped pair (a, b), the same argument applies: a takes the role of c', b the role of d', and the relevant ordering is any ordering with agent 1 placed second. Thus, g satisfies SP.

It remains to show that $f \neq f_{\mathrm{RSD}}$. Since RSD satisfies anonymity, we have $\pi(f_{\mathrm{RSD}}) = f_{\mathrm{RSD}}$ for every $\pi \in \Pi$. Fix a profile $\mathbf{P} \in \mathsf{P}$. By the definition of g, $g(\mathbf{P}) \neq f_{\mathrm{RSD}}(\mathbf{P})$. Moreover, for every $\mathrm{id} \neq \pi \in \Pi$, we have $\pi(\mathbf{P}) \notin \mathsf{P}$: Indeed, agent 1 is the only agent whose top two houses in $H \setminus \{e\}$ are $\{a, b\}$, and agents $2, 3, 4$ each have a distinct second choice. Consequently, $\pi(g)(\mathbf{P}) = \pi(f_{\mathrm{RSD}})(\mathbf{P})$. Since f is the average of $\{\pi(g)\}_{\pi \in \Pi}$ and f_{RSD} is the average of $\{\pi(f_{\mathrm{RSD}})\}_{\pi \in \Pi}$, it follows that $f(\mathbf{P}) \neq f_{\mathrm{RSD}}(\mathbf{P})$ and therefore $f \neq f_{\mathrm{RSD}}$. Thus, the axioms do not suffice to characterize f_{RSD} in these settings. ■

Remark 12 (Dominance over f_{RSD}). The mechanism constructed in the proof of Proposition 5.2 does not merely differ from f_{RSD}; it dominates it. Concretely, for every profile, it weakly improves each agent's allocation relative to f_{RSD}; moreover, at some profile, it strictly improves at least one agent's allocation.

This follows directly from the construction in the proof: relative to f_{RSD}, the only change is that at certain profiles, it shifts an ε-amount of probability mass in one agent's allocation from house x to house y, which this agent ranks strictly above x, while leaving all other agents' allocations unchanged. The subsequent symmetrization step (averaging over renamings) preserves these profile-by-profile weak improvements for every agent and therefore yields strict improvement at any profile where at least one of the averaged terms performs the ε-shift.

Remark 13. The dominance over f_{RSD} established in Remark 12 is not confined to the specific market sizes treated in Proposition 5.2. Using the same extension-and-symmetrization idea described in Remark 8, one can lift the construction to any (n, m) market with $4 \leq n \leq m - 1$. Briefly, one enlarges the market by adding an equal number of auxiliary agents and new houses, and then lets the auxiliary agents select first. If, after the auxiliary agents select, the set of remaining houses coincides with the original set of houses (that is, none of the original houses is selected by an auxiliary agent), then one applies the dominating mechanism constructed in Proposition 5.2; otherwise, one applies f_{RSD}. Finally, one uniformly averages over all renamings of the agents. The resulting mechanism satisfies the axioms and dominates f_{RSD} in the (n, m) market. We omit the details.

Remark 14. The non-uniqueness results naturally raise a welfare question: in domains where the axioms do not uniquely characterize f_{RSD}, can f_{RSD} be dominated by another mechanism satisfying the same axioms? Remarks 8 and 13 provide an affirmative answer to this question for $n \geq m + 2$ (where $m \geq 2$) and for $4 \leq n \leq m - 1$, respectively. In the remaining regions of non-uniqueness, namely $n = m + 1 \geq 4$ and $n = m \geq 5$, it is an open question whether dominance is possible, or whether f_{RSD} is undominated despite non-uniqueness.

5.3 Non-uniqueness under strengthened and additional axioms

Since we have established that f_{RSD} is not always the only mechanism satisfying ExPE, ETE, and SP, a natural question arises: can adding or strengthening axioms guarantee uniqueness for every number of agents n and houses m? Basteck and Ehlers [5] have raised this type of question with respect to adding the axiom *Bounded Invariance (BI)*:

Definition 5.1. A mechanism f satisfies *Bounded Invariance (BI)* if for every preference profile $\mathbf{P} \in \mathcal{R}^N$, agent $i \in N$, preference order $P_i' \in \mathcal{R}$, and house $h \in H$, the following holds:

$$\left(\forall h' \in C_{P_i}(h) : C_{P_i'}(h') = C_{P_i}(h')\right) \implies f(\mathbf{P})_h = f(\mathbf{P}_{-i}, P_i')_h .$$

In other words, if an agent changes its ranking without changing the part above and including h, then the assignment probability of h remains unchanged for every agent.

However, the answer to the uniqueness question under this extra axiom is negative, as the following result shows, even when strengthening ETE to anonymity and adding neutrality.

Proposition 5.3. When $n \geq m-1 \geq 5$, f_{RSD} is not the unique mechanism satisfying anonymity, neutrality, ExPE, SP, and BI.

Proof. We construct a suitable mechanism f. Denote $H := \{a, b, h_1, \ldots, h_{m-2}\}$ and $j := m-1 \in N$; from this point on, $m-1$ refers to the integer, and "agent j" refers to the agent with index $m-1$.

Define

$$\mathsf{P} := \left\{\mathbf{P} \in \mathcal{R}^N \mid \forall i \in [m-2], \forall h \in H \setminus \{a, b, h_i\} : aP_ibP_ih_iP_ih\right\}.$$

Thus, for each $i \in [m-2]$, agent i's top three choices are a, b, and h_i, in that order. For every $\mathbf{P} \in \mathcal{R}^N$, let $x := x_{\mathbf{P}}$ and $y := y_{\mathbf{P}}$ denote agent j's second and third choices among $H \setminus \{a, b\}$, respectively. Fix $\varepsilon > 0$ sufficiently small (e.g., $\varepsilon \leq \frac{1}{n!}$), and define $v := (v_h)_{h \in H}$ by

$$v_h := \begin{cases} \varepsilon & \text{if } h = x, \\ -\varepsilon & \text{if } h = y, \\ 0 & \text{otherwise.} \end{cases}$$

With v as defined, we specify the mechanism g by altering f_{RSD} only on P: shift ε probability from y to x in agent j's assignment and, if $n \geq m$, make the opposite shift for agent m; namely,

$$g(\mathbf{P})_i = \begin{cases} f_{\mathrm{RSD}}(\mathbf{P})_i + v & \text{if } \mathbf{P} \in \mathsf{P} \text{ and } i = j, \\ f_{\mathrm{RSD}}(\mathbf{P})_i - v & \text{if } \mathbf{P} \in \mathsf{P} \text{ and } i = m \text{ (when } n \geq m), \\ f_{\mathrm{RSD}}(\mathbf{P})_i & \text{otherwise.} \end{cases}$$

From now on, we assume without loss of generality that $h_i P_j h_{i+1}$ for every $i \in [m-3]$; the remaining cases follow by permuting the roles of agents $1, \ldots, m-2$ consistently with the permutation of agent j's preferences among $h_1, \ldots, h_{m-2}$. Thus, $x_{\mathbf{P}} = h_2$ and $y_{\mathbf{P}} = h_3$.

To show that g is well defined, it suffices to verify that, under f_{RSD}, each of the following occurs with probability at least ε: agent j receives h_3; when $n = m-1$, the house h_2 remains unassigned; and when $n \geq m$, agent m receives h_2. Since each ordering in RSD has probability $\frac{1}{n!}$ and $\varepsilon \leq \frac{1}{n!}$, it is enough to exhibit one ordering for each event. For agent j: in an ordering starting with $4321j$, the first four agents take a, b, h_2, h_1, so when agent j arrives he receives h_3. For the remaining events, consider an ordering that begins with $21j$, followed by the agents in $\{3, \ldots, m-2\}$ in some order; here agents 2 and 1 take a and b, agent j then takes h_1, and the agents in $\{3, \ldots, m-2\}$ take their respective h_i's, leaving h_2 as the only unassigned house. If $n = m-1$, this already lists all agents, so the ordering is complete and h_2 remains unassigned; if $n \geq m$, let agent m come next and he receives h_2. Therefore, g is well defined.

Now define the mechanism f as the symmetrization of g over all renamings of agents and houses:

$$f = \frac{1}{n!m!} \sum_{(\pi,\tau) \in \Pi \times \Gamma} (\pi, \tau)(g).$$

Note also that, since f_{RSD} is anonymous and neutral, it equals its own symmetrization:

$$f_{\text{RSD}} = \frac{1}{n!m!} \sum_{(\pi,\tau) \in \Pi \times \Gamma} (\pi, \tau)(f_{\text{RSD}}).$$

We claim that f is the desired mechanism. By construction, f is anonymous and neutral. Moreover, ExPE, SP, and BI are invariant under renamings and are preserved under convex combinations; hence it suffices to verify these properties for g. We will therefore show that g satisfies ExPE, SP, and BI, and we will also establish that $f \neq f_{\text{RSD}}$.

We now verify that g satisfies ExPE. We note that f_{RSD} satisfies ExPE; hence, it suffices to consider $\mathbf{P} \in \mathsf{P}$ and to show that the adjustment from f_{RSD} to g can be realized as a probability shift between efficient assignments, reducing at most ε probability from any single assignment.

Consider the four assignments s_1, s_2, s'_1, s'_2 given by

$$s_1(i) = \begin{cases} a & \text{if } i = 4, \\ b & \text{if } i = 3, \\ h_3 & \text{if } i = j, \\ h_i & \text{if } i \in [m-2] \setminus \{3,4\}, \\ h_4 & \text{if } i = m, \\ \varnothing & \text{otherwise,} \end{cases} \quad s_2(i) = \begin{cases} a & \text{if } i = 2, \\ b & \text{if } i = 1, \\ h_1 & \text{if } i = j, \\ h_i & \text{if } i \in [m-2] \setminus \{1,2\}, \\ h_2 & \text{if } i = m, \\ \varnothing & \text{otherwise,} \end{cases}$$

$$s'_1(i) = \begin{cases} a & \text{if } i = 4, \\ b & \text{if } i = 1, \\ h_1 & \text{if } i = j, \\ h_i & \text{if } i \in [m-2] \setminus \{1,4\}, \\ h_4 & \text{if } i = m, \\ \varnothing & \text{otherwise,} \end{cases} \quad s'_2(i) = \begin{cases} a & \text{if } i = 2, \\ b & \text{if } i = 3, \\ h_2 & \text{if } i = j, \\ h_i & \text{if } i \in [m-2] \setminus \{2,3\}, \\ h_3 & \text{if } i = m, \\ \varnothing & \text{otherwise.} \end{cases}$$

Reducing ε probability from s_1 and s_2 and adding ε probability to s'_1 and s'_2 implements exactly the required adjustment: agent j shifts ε probability from $y = h_3$ to $x = h_2$; when $n \geq m$, agent m shifts ε probability from $x = h_2$ to $y = h_3$; and all other agents' marginals are unchanged. Each of s_1, s_2, s'_1, s'_2 is the outcome of some ordering of the agents; hence each is an efficient assignment. For example, they can be obtained from orderings whose initial segments are $4321j$, $21j34$, $41j23$, and $2314j$, respectively; in each case the ordering then continues with the remaining agents in $[m-2]$ in some order, then (if $n \geq m$) agent m, and finally the rest. Hence the adjustment is a probability shift between efficient assignments, and g satisfies ExPE.

For SP, since f_{RSD} satisfies it, every agent in $N \setminus \{j, m\}$ cannot gain by manipulation, because their assignments coincide under g and f_{RSD} at every profile. Moreover, agent m (if he exists) cannot gain either, since his report cannot affect the adjustment. It remains to show that agent j cannot gain.

Let $\mathbf{P} \in \mathcal{R}^N$. As before, it suffices to consider manipulations by agent j that swap adjacent houses and thereby change the adjustment. If $\mathbf{P} \notin \mathsf{P}$, then $(\mathbf{P}_{-j}, R_j) \notin \mathsf{P}$ for all $R_j \in \mathcal{R}$, so the adjustment never changes. For $\mathbf{P} \in \mathsf{P}$, the only adjacent swaps by agent j that can affect the adjustment are between the pairs (h_k, h_{k+1}) for $k = 1, 2, 3$. By the definition of g, the difference between the adjustments to agent j's assignment before and after any such swap consists solely of a probability shift between the two houses in the swapped pair.

- If the swapped pair is (h_2, h_3), the adjustment flips from transferring ε probability from h_3 to h_2 to the opposite transfer. This is worse for agent j, so he cannot gain from this swap.
- If the swapped pair is (h_3, h_4), the adjustment changes from transferring ε probability from h_3 to h_2 to transferring ε from h_4 to h_2. Equivalently, the net change shifts ε probability from h_4 to h_3. By the same reasoning as in Proposition 5.2, it suffices to exhibit some ordering in which agent j receives a house in $C_{P_j}(h_3)$ before the swap but not after it. Any ordering whose initial segment is $4321j$ has this property: under $\mathbf{P}$, agent j receives h_3,

whereas after the swap he receives h_4.

- If the swapped pair is (h_1, h_2), the same argument applies, now taking some ordering whose initial segment is $21j$.

Therefore, g satisfies SP.

For BI, let $\mathbf{P} \in \mathcal{R}^N$, $i \in N$, $P'_i \in \mathcal{R}$, and $h \in H$. Any change from P_i to P'_i can be written as a sequence of adjacent swaps. Since BI requires equality at each step for every house that appears above the swap at that step, it suffices to consider the case where P'_i differs from P_i by a single adjacent swap. Because f_{RSD} satisfies BI, if the adjustments for $\mathbf{P}$ and $\mathbf{P}' := (\mathbf{P}_{-i}, P'_i)$ are identical with respect to h, then

$$g(\mathbf{P}')_h - g(\mathbf{P})_h = f_{\text{RSD}}(\mathbf{P}')_h - f_{\text{RSD}}(\mathbf{P})_h = 0.$$

Thus it remains to check whether there exists a swap below a house h that changes the adjustment at h.

If $i \in N \setminus [m-1]$, then no swap by agent i changes the adjustment. If $i \in [m-2]$, the only houses that can lie above a swap of agent i that changes the adjustment are a and b; however, the adjustment never alters any agent's probability of receiving a or b. Otherwise, $i = j$. In that case, as noted earlier, the only adjacent swaps of agent j that can affect the adjustment are between the pairs (h_k, h_{k+1}) for $k = 1, 2, 3$. Each such swap shifts probability only between the two swapped houses for agent j (and, if $n \geq m$, for agent m as well). Hence no probability associated with any house above the swap is affected. Therefore, g satisfies BI. Hence, f satisfies all the axioms stated in the proposition.

It remains to show that $f \neq f_{\text{RSD}}$. Fix a profile $\mathbf{Q} \in \mathsf{P}$ such that: agent j's top choices are $h_1 Q_j h_2$; for every $i \in N \setminus \{j\}$, agent i's top two choices are $a Q_i b$; and for every $i \in N \setminus \{2, j\}$, the house h_2 is not among i's top four choices (these conditions are feasible since $m \geq 6$). By the construction of g, we have $g(\mathbf{Q})_{h_2,j} > f_{\text{RSD}}(\mathbf{Q})_{h_2,j}$.

We claim that for every $(\pi, \tau) \in \Pi \times \Gamma$,

$$(\pi, \tau)(g)(\mathbf{Q})_{h_2,j} \geq (\pi, \tau)(f_{\text{RSD}})(\mathbf{Q})_{h_2,j}.$$

Assume, toward a contradiction, that this fails for some (π, τ). Let $\mathbf{Q}' := (\pi, \tau)(\mathbf{Q})$. Then

$$g(\mathbf{Q}')_{\tau(h_2),\pi(j)} < f_{\text{RSD}}(\mathbf{Q}')_{\tau(h_2),\pi(j)}.$$

Note that $\tau(h_2)$ is the second most-preferred house of $\pi(j)$ in $\mathbf{Q}'$. The only agent whose probability of obtaining his second most-preferred house can decrease under the adjustment from f_{RSD} to g is agent m, hence $\pi(j) = m$.

Such a decrease can occur only if agent j ranks $\tau(h_2)$ second among $H \setminus \{a, b\}$ in $\mathbf{Q}'$. Consequently, agent j must rank $\tau(h_2)$ among his top four houses in $\mathbf{Q}'$, so agent $\pi^{-1}(j)$ must rank h_2 among his top four houses in $\mathbf{Q}$. By the construction of $\mathbf{Q}$, this forces $\pi^{-1}(j) \in \{2, j\}$; since $\pi(j) \neq j$, we have $\pi^{-1}(j) = 2$.

In $\mathbf{Q}$, the top three choices of agent 2 are $a Q_2 b Q_2 h_2$; therefore, in $\mathbf{Q}'$ the top three choices of $\pi(2) = j$ are $\tau(a) Q'_j \tau(b) Q'_j \tau(h_2)$. Because j ranks $\tau(h_2)$ second among $H \setminus \{a, b\}$ in $\mathbf{Q}'$, we must have $\{\tau(a), \tau(b)\} \neq \{a, b\}$.

Since $\mathbf{Q}' \in \mathsf{P}$ (otherwise no adjustment would occur), agents $1, \ldots, m-2$ must rank a and b as their top two houses in $\mathbf{Q}'$. Hence agents $\pi^{-1}(1), \ldots, \pi^{-1}(m-2)$ rank $\tau^{-1}(a)$ and $\tau^{-1}(b)$ as their top two houses in $\mathbf{Q}$. We obtain a contradiction: in $\mathbf{Q}$ every agent $i \neq j$ has $\{a, b\}$ as his top two houses, and since $\{\tau^{-1}(a), \tau^{-1}(b)\} \neq \{a, b\}$, only j could have $\{\tau^{-1}(a), \tau^{-1}(b)\}$ as his top two. Yet $\pi^{-1}(1), \ldots, \pi^{-1}(m-2)$ are $m-2 \geq 4$ distinct agents, so they cannot all be j. Therefore the claimed inequality holds for every $(\pi, \tau) \in \Pi \times \Gamma$.

Finally, since f is the average of $\{(\pi, \tau)(g)\}_{(\pi,\tau)\in\Pi\times\Gamma}$ and f_{RSD} is the average of $\{(\pi, \tau)(f_{\text{RSD}})\}_{(\pi,\tau)\in\Pi\times\Gamma}$, we obtain

$$f(\mathbf{Q})_{h_2,j} > f_{\text{RSD}}(\mathbf{Q})_{h_2,j},$$

and hence $f \neq f_{\text{RSD}}$, as desired. ■

A Adding more axioms: persistence of non-uniqueness

Section 5.3 established that anonymity, neutrality, ExPE, SP and BI do not uniquely characterize f_{RSD}. To pursue uniqueness, we introduce two additional axioms, both satisfied by f_{RSD}, and examine whether the extended set of axioms suffices for uniqueness. However, this attempt fails: the mechanism from Proposition 5.3 also satisfies one of these new axioms, and under a slightly stricter size condition, $n \geq m-1 \geq 7$ (compared to $n \geq m-1 \geq 5$ in Proposition 5.3), we construct another mechanism, similar in spirit to that of Proposition 5.3, which satisfies all the axioms (those from the proposition together with the two new ones) and still differs from f_{RSD}. We now define one of these additional axioms.

Definition A.1 (NB)**.** A mechanism f satisfies *Non-Bossiness (NB)* if, for every preference profile $\mathbf{P} \in \mathcal{R}^N$, agent $i \in N$, and preference order $P_i' \in \mathcal{R}$,

$$f(\mathbf{P})_i = f(\mathbf{P}_{-i}, P_i')_i \implies f(\mathbf{P}) = f(\mathbf{P}_{-i}, P_i').$$

In words, an agent cannot alter any other agent's assignment without simultaneously changing his own. The mechanism f_{RSD} satisfies NB (see [2]). Before showing that the mechanism from Proposition 5.3 satisfies this axiom, we first establish an auxiliary claim that will be useful in the proof. The claim shows that for mechanisms satisfying SP, it is sufficient to verify NB only for preference orders that differ by an adjacent swap.

Claim A.1. Let f be a mechanism that satisfies SP. Suppose that for every preference profile $\mathbf{P} \in \mathcal{R}^N$, agent $i \in N$, and preference order $P_i'' \in \mathcal{R}$ obtained from P_i by an adjacent swap,

$$f(\mathbf{P})_i = f(\mathbf{P}_{-i}, P_i'')_i \implies f(\mathbf{P}) = f(\mathbf{P}_{-i}, P_i'').$$

Then f satisfies NB.

Proof. Fix $i \in N$. For every $P_i, P_i' \in \mathcal{R}$, denote by $\left(R_i^\ell\right)_{\ell=1}^k$ the sequence of preference orders satisfying $R_i^1 = P_i$, $R_i^k = P_i'$, and for every $1 \leq \ell < k$, $R_i^{\ell+1}$ is obtained from R_i^ℓ by an adjacent swap that elevates the house h_ℓ, defined as the house satisfying $C_{R_i^\ell}(h_\ell) \neq C_{P_i'}(h_\ell)$ and, for every $h \in C_{P_i'}(h_\ell) \setminus \{h_\ell\}$, $C_{R_i^\ell}(h) = C_{P_i'}(h)$.

We show that for every $\mathbf{P} \in \mathcal{R}^N$ and $P_i' \in \mathcal{R}$ such that $f(\mathbf{P})_i = f(\mathbf{P}_{-i}, P_i')_i$, it also holds that $f(\mathbf{P}) = f(\mathbf{P}_{-i}, P_i')$, by induction on k (the length of the sequence from P_i to P_i'). The base case $k = 1$ is immediate, since $P_i' = P_i$. For the inductive step, assume the claim holds for all sequences of length smaller than k. Denote $P_i'' \coloneqq R_i^2$. Note that h_1 is a house that does not move downward in any of the adjacent swaps corresponding to the sequence from P_i to P_i'. Since f satisfies SP, we have

$$f(\mathbf{P})_{h_1,i} \leq f(\mathbf{P}_{-i}, P_i'')_{h_1,i} \leq f(\mathbf{P}_{-i}, P_i')_{h_1,i}.$$

Because $f(\mathbf{P})_i = f(\mathbf{P}_{-i}, P_i')_i$, these inequalities must in fact be equalities. Moreover, since f satisfies SP, this implies $f(\mathbf{P})_i = f(\mathbf{P}_{-i}, P_i'')_i$. By the assumption of the claim, we then have $f(\mathbf{P}) = f(\mathbf{P}_{-i}, P_i'')$, and since the sequence from P_i'' to P_i' has length less than k, the induction hypothesis gives $f(\mathbf{P}_{-i}, P_i'') = f(\mathbf{P}_{-i}, P_i')$. Combining these equalities yields the desired result. ■

We now show that the mechanism from Proposition 5.3 also satisfies this axiom.

Claim A.2. The mechanism f from Proposition 5.3 satisfies NB (for small enough ε).

Proof. Let $\mathbf{P} \in \mathcal{R}^N$ and $i \in N$. Since f satisfies SP, the previous claim implies that it is sufficient to show that

$$f(\mathbf{P})_i = f(\mathbf{P}_{-i}, P_i')_i \implies f(\mathbf{P}) = f(\mathbf{P}_{-i}, P_i')$$

for every preference order P_i' that can be obtained from P_i by an adjacent swap. Let $P_i' \in \mathcal{R}$ be such an order and denote $\mathbf{P}' \coloneqq (\mathbf{P}_{-i}, P_i')$. We will show that if $f(\mathbf{P})_i = f(\mathbf{P}')_i$, then $f(\mathbf{P}) = f(\mathbf{P}')$. Recall that f_{RSD} satisfies NB [2]. Define $d \coloneqq g - f_{\text{RSD}}$ (that is, $d(\mathbf{P}) = g(\mathbf{P}) - f_{\text{RSD}}(\mathbf{P})$ for every $\mathbf{P} \in \mathcal{R}^N$) and $D \coloneqq f - f_{\text{RSD}}$. Since f is the symmetrization of g (and f_{RSD} is the symmetrization of itself), D is the symmetrization of d, i.e.

$$D = \frac{1}{n!m!} \sum_{(\pi,\tau) \in \Pi \times \Gamma} (\pi, \tau)(d).$$

Here, the notation $(\pi, \tau)(d)$ is used in the same sense as for mechanisms, even though d is not itself a mechanism; specifically, $(\pi, \tau)(d)$ denotes the difference $(\pi, \tau)(g) - (\pi, \tau)(f_{\text{RSD}})$. Moreover, by the definition of g, for every $\mathbf{P} \in \mathcal{R}^N$, $i \in N$, and $h \in H$, $\left|d(\mathbf{P})_{h,i}\right| \leq \varepsilon$, and we assume here that $\varepsilon < \frac{1}{2n!}$.

We first show that if $f(\mathbf{P})_i = f(\mathbf{P}')_i$, then $f_{\text{RSD}}(\mathbf{P})_i = f_{\text{RSD}}(\mathbf{P}')_i$. Assume, for contradiction, that this is not the case. Then, there exists $h \in H$ such that $f_{\text{RSD}}(\mathbf{P})_{h,i} \neq f_{\text{RSD}}(\mathbf{P}')_{h,i}$. Since f_{RSD} assigns each deterministic SD mechanism a probability of $\frac{1}{n!}$, it follows that

$$\left|f_{\text{RSD}}(\mathbf{P})_{h,i} - f_{\text{RSD}}(\mathbf{P}')_{h,i}\right| \geq \frac{1}{n!}.$$

By the definition of D, this implies

$$\left|D\left(\mathbf{P}\right)_{h,i}-D\left(\mathbf{P}'\right)_{h,i}\right|\geq\frac{1}{n!}.$$

On the other hand,

$$\begin{aligned}\left|D\left(\mathbf{P}\right)_{h,i}-D\left(\mathbf{P}'\right)_{h,i}\right| &\leq\frac{1}{n!m!}\sum_{(\pi,\tau)\in\Pi\times\Gamma}\left|d\left(\left(\pi,\tau\right)\left(\mathbf{P}\right)\right)_{\tau(h),\pi(i)}\right|+\left|d\left(\left(\pi,\tau\right)\left(\mathbf{P}'\right)\right)_{\tau(h),\pi(i)}\right|\\ &\leq\frac{1}{n!m!}\sum_{(\pi,\tau)\in\Pi\times\Gamma}2\varepsilon\\ &<\frac{1}{n!}\end{aligned}$$

which contradicts the previous inequality. Therefore, $f_{\text{RSD}}\left(\mathbf{P}\right)_i=f_{\text{RSD}}\left(\mathbf{P}'\right)_i$.

Since f_{RSD} satisfies NB, we also have $f_{\text{RSD}}\left(\mathbf{P}\right)=f_{\text{RSD}}\left(\mathbf{P}'\right)$. Thus, to establish $f\left(\mathbf{P}\right)=f\left(\mathbf{P}'\right)$, it suffices to show that $D\left(\mathbf{P}\right)=D\left(\mathbf{P}'\right)$. Assume, for contradiction, that this is not the case. Because $f\left(\mathbf{P}\right)_i=f\left(\mathbf{P}'\right)_i$ and $f_{\text{RSD}}\left(\mathbf{P}\right)_i=f_{\text{RSD}}\left(\mathbf{P}'\right)_i$, we obtain $D\left(\mathbf{P}\right)_i=D\left(\mathbf{P}'\right)_i$. Hence, there exists an agent $k\in N\setminus\{i\}$ and a house $h\in H$ such that $D\left(\mathbf{P}\right)_{h,k}\neq D\left(\mathbf{P}'\right)_{h,k}$. Since D is the symmetrization of d, there must exist some $(\pi,\tau)\in\Pi\times\Gamma$ such that $\left(\pi,\tau\right)\left(d\right)\left(\mathbf{P}\right)_{h,k}\neq\left(\pi,\tau\right)\left(d\right)\left(\mathbf{P}'\right)_{h,k}$.

Without loss of generality, we may assume $(\pi,\tau)=(\text{id},\text{id})$; otherwise, we can replace $\mathbf{P},\mathbf{P}',i,k,h$ with $\left(\pi,\tau\right)\left(\mathbf{P}\right),\left(\pi,\tau\right)\left(\mathbf{P}'\right),\pi\left(i\right),\pi\left(k\right),\tau\left(h\right)$ and apply the same argument. Thus, $d\left(\mathbf{P}\right)_{h,k}\neq d\left(\mathbf{P}'\right)_{h,k}$. We will now show that this contradicts the earlier conclusion that $f_{\text{RSD}}\left(\mathbf{P}\right)_i=f_{\text{RSD}}\left(\mathbf{P}'\right)_i$.

Since agents in $N\setminus[m-1]$ cannot influence the adjustment by changing their preferences, we may assume $i\in[m-1]$. Moreover, when agent i performs an adjacent swap that elevates some $h'\in H$ in his ranking, the set of orderings where agent i receives h' in the post-swap profile contains the corresponding set before the swap. Hence, to establish $f_{\text{RSD}}\left(\mathbf{P}\right)_{h',i}\neq f_{\text{RSD}}\left(\mathbf{P}'\right)_{h',i}$, it suffices to exhibit an ordering in which agent i receives h' only after the swap.

If $i\in[m-2]$, the swap in agent i's preference order can affect the adjustment only when exactly one of $\mathbf{P}$ or $\mathbf{P}'$ belongs to P. Without loss of generality, suppose $\mathbf{P}\in\mathsf{P}$. Then, the top three houses in P_i are $aP_ibP_ih_i$, and the swap involves one of the following adjacent pairs: (a,b), (b,h_i), or (h_i,h) for some $h\in H\setminus\{a,b,h_i\}$. Let $i',i''\in[m-2]\setminus\{i\}$ be two distinct agents. For each of the adjacent pairs mentioned, there exists an ordering where agent i receives the elevated house only after the swap: if the pair is (a,b), let i go first; if (b,h_i), start with $i'i$; and if (h_i,h) for some $h\in H\setminus\{a,b,h_i\}$, start with $i''i'i$.

Otherwise, $i=j$. In this case, the swap in agent j's preference order can affect the adjustment only when both $\mathbf{P}$ and $\mathbf{P}'$ belong to P. Without loss of generality, assume $h_\ell P_j h_{\ell+1}$ for every $\ell\in[m-3]$; then the swap must involve one of the adjacent pairs (h_1,h_2), (h_2,h_3), or (h_3,h_4). For each of these pairs, there exists an ordering where agent j receives the elevated house only after the swap: for (h_1,h_2), consider an ordering starting with $21j$; for (h_2,h_3), an ordering starting with $321j$; and for (h_3,h_4), an ordering starting with $4321j$.

In all cases, we therefore obtain $f_{\text{RSD}}\left(\mathbf{P}\right)_i\neq f_{\text{RSD}}\left(\mathbf{P}'\right)_i$. This contradiction implies that our assumption was false, and hence $D(\mathbf{P}) = D(\mathbf{P}')$. Therefore, $f(\mathbf{P}) = f(\mathbf{P}')$, and f satisfies NB, as required. ■

We now introduce the second new axiom.

Definition A.2 (CM)**.** A mechanism f satisfies *Cross Monotonicity (CM)* if, for every preference profile $\mathbf{P} \in \mathcal{R}^N$, agent $i \in N$, pair of adjacent houses $h' P_i^+ h$, and agent $j \in N \setminus \{i\}$,

$$f\left(\mathbf{P}_{-i}, P_i^h\right)_{h,j} \leq f(\mathbf{P})_{h,j},$$

where P_i^h is the preference order obtained from P_i by swapping h with h'.

In words, if agent i swaps house h with the house immediately above it in his preference order, then every other agent's probability of receiving h weakly decreases. To the best of our knowledge, the CM axiom has not appeared in the literature on assignment mechanisms, though a similar notion has been studied in the context of cost sharing [12]. We next observe that f_{RSD} satisfies the CM axiom, as stated in the following claim.

Claim A.3. f_{RSD} satisfies CM.

Proof. Since the distribution of RSD over the SD mechanisms does not depend on the preference profile, it suffices to show that for every two distinct agents $i, i' \in N$, profile $\mathbf{P} \in \mathcal{R}^N$, and pair of adjacent houses $h P_i^+ h'$, the set of orderings in which i' receives h under $\mathbf{P}$ is contained in the corresponding set under the profile obtained from $\mathbf{P}$ after agent i swaps h with h'.

Indeed, if i' appears before i, a change in i's preferences does not affect i''s outcome. If i' receives h when arriving after i under the profile $\mathbf{P}$, then h must have been available to i, but i chose some other house h'' such that $h'' P_i^+ h$. Agent i would make the same choice after swapping h with h', and therefore i' continues to receive h in all such orderings, as required. ■

Having defined the new axioms, we now show that there exists a mechanism other than f_{RSD} satisfying all of them.

Proposition A.1. When $n \geq m-1 \geq 7$, f_{RSD} is not the unique mechanism satisfying anonymity, neutrality, ExPE, SP, BI, NB, and CM.

Proof. We construct a suitable mechanism f. As in Proposition 5.3, let $H \coloneqq \{a, b, h_1, \ldots, h_{m-2}\}$, and let $j \in N$ denote the agent with index $m-1$. Define P exactly as before, and let $x = x_{\mathbf{P}}$ and $y = y_{\mathbf{P}}$ denote agent j's third and fourth choices among $H \setminus \{a, b\}$, respectively.

Fix $\varepsilon > 0$ sufficiently small (as in the previous claim, we assume $\varepsilon < \frac{1}{2n!}$), and define the vector $v = (v_h)_{h \in H}$ and the mechanism g as in Proposition 5.3. We can again assume, without loss of generality, that $h_i P_j h_{i+1}$ for every $i \in [m-3]$, so that $x_{\mathbf{P}} = h_3$ and $y_{\mathbf{P}} = h_4$.

To verify that g is well defined, we apply the same reasoning as in Proposition 5.3. In this case, agent j receives h_4 in an ordering starting with $54321j$. In an ordering beginning with $321j$, followed by the agents in $\{4, \ldots, m-2\}$ in some order, h_3 remains the only unassigned house afterward. Hence, h_3 remains unassigned when $n = m-1$; and when $n \geq m$, we can place agent m immediately after these agents, assigning him h_3. Therefore, g is well defined.

Define f as the symmetrization of g over all renamings of agents and houses. We show that f is the desired mechanism. By construction, f satisfies anonymity and neutrality. For ExPE, SP, BI, and CM, it suffices to verify that g satisfies them. The arguments for the first three are similar to those in Proposition 5.3, and we highlight below the necessary modifications that allow the reasoning to be adapted to the present setting. For BI, we use the same reasoning as before, noting that the three adjacent pairs of houses in j's preference order that influence the adjustment are (h_k, h_{k+1}) for $k = 2, 3, 4$. For SP, the pairs of adjacent houses that modify the adjustment in the wrong direction (i.e., in a way favorable to agent j) are (h_2, h_3) and (h_4, h_5). As before, it suffices to exhibit an ordering where agent j receives a house in $C_{P_j}(h_2)$ before the swap of h_2 and h_3 but not afterward, and an ordering where he receives a house in $C_{P_j}(h_4)$ before the swap of h_4 and h_5 but not afterward. Two such orderings are those whose initial segments are $321j$ and $54321j$.

For ExPE, the adjustment can be described by a transfer of probabilities that decreases ε from each of s_1 and s_2 and adds ε to each of s'_1 and s'_2, where the explicit form of these assignments is given below.

$$s_1(i) = \begin{cases} a & i = 5, \\ b & i = 4, \\ h_4 & i = j, \\ h_i & i \in [m-2] \setminus \{4,5\}, \\ h_5 & i = m, \\ \varnothing & \text{otherwise}, \end{cases} \quad s_2(i) = \begin{cases} a & i = 3, \\ b & i = 1, \\ h_1 & i = j, \\ h_i & i \in [m-2] \setminus \{1,3\}, \\ h_3 & i = m, \\ \varnothing & \text{otherwise}, \end{cases}$$

$$s'_1(i) = \begin{cases} a & i = 5, \\ b & i = 1, \\ h_1 & i = j, \\ h_i & i \in [m-2] \setminus \{1,5\}, \\ h_5 & i = m, \\ \varnothing & \text{otherwise}, \end{cases} \quad s'_2(i) = \begin{cases} a & i = 3, \\ b & i = 4, \\ h_3 & i = j, \\ h_i & i \in [m-2] \setminus \{3,4\}, \\ h_4 & i = m, \\ \varnothing & \text{otherwise}. \end{cases}$$

These assignments are efficient, as they correspond to orderings beginning with $54321j$, $31j$, $51j$, $3412j$, followed by the remaining agents in $[m-2]$ in some order, and finally (if $n \geq m$) agent m, who receives the remaining house.

Since f satisfies SP and has a structure similar to that of the mechanism in Proposition 5.3, the same reasoning as in the previous claim shows that f also satisfies NB.

We next show that g satisfies CM. Because f_{RSD} satisfies CM, using arguments analogous to those employed to establish SP and BI, it suffices to focus on adjacent swaps that influence the adjustment. For each such swap, let i denote the agent performing the swap, h the house that moves up in i's preference order, and i' another agent whose adjustment changes in the wrong direction, that is, in a way that increases the probability that i' receives h when i raises h in his preference order. For each such agent i', it is enough to exhibit an ordering that overrides this change, namely one in which i' receives h before the swap but not after i raises it in his preference order. This will suffice, because the set of orderings in which i' receives h when i ranks it higher is contained in the corresponding set of orderings when i ranks it lower, by the same reasoning as in the previous claim.

Observe that only agents in $[m-1]$ can make an adjacent swap that affects the adjustment, and this occurs only when either the profile before the swap or the one after it belongs to P. By considering the opposite inequality for the house that moves down in the swap, we may assume without loss of generality that the profile before the swap belongs to P; we denote this profile by $\mathbf{P}$. Moreover, the only agents affected by the adjustment are j and m (if the latter exists), and since the adjustment for these two agents always changes in opposite directions, it cannot be in the undesired direction for both simultaneously. Hence, for every swap we consider, there is at most one agent (either j or m) for whom the change is in the undesired direction, and for that agent we will need to exhibit an ordering that overrides this change. We distinguish between cases according to the agent performing the swap.

- Agent j: In this case, we examine only the adjustment of agent m. As noted earlier, the only swaps that agent j can make that affect the adjustment are between the adjacent pairs (h_k, h_{k+1}) for $k = 2, 3, 4$.
  - When $k = 3$ (that is, j changes his preferences to $P_j^{h_4}$), the change in the adjustment for agent m is in the correct direction for both houses: a negative change for h_4 and a positive one for h_3.
  - When $k \in \{2, 4\}$, the change in the adjustment for agent m is in the undesired direction for both houses. Consider an ordering in which agents k and $k+1$ appear first, followed by the remaining agents in $[m-2]$, then agent j, and finally agent m. In this ordering, agent m receives the house that agent j prefers less among h_k and h_{k+1}. This construction simultaneously demonstrates that there exists an ordering where agent m receives h_{k+1} before the swap but not afterward, and an ordering where he does not receive h_k before the swap but does receive it afterward, as required.
- Agents in $[m-2]$: a swap performed by these agents affects the adjustment only if it involves one of their top three houses. The only houses for which the adjustment changes are h_3 and h_4; thus, at least one of them must participate in the swap. We analyze the cases for each house separately.
  - When h_3 is involved in the swap, we distinguish cases according to the identity of the swapping agent.
    - When the swapping agent is $i = 3$: house h_3 can be swapped either upward (with b) or downward (with some h_k for $k \in [m-2] \setminus \{3\}$).
      - If h_3 is swapped with b, the change in the adjustment is in the wrong direction for agent m. We therefore need an ordering where agent m receives h_3 before the swap but not afterward. An ordering beginning with $13j$, followed by the other agents in $[m-2]$ in some order, and then agent m, satisfies this

condition.

        - If h_3 is swapped downward with some h_k for $k \in [m-2] \setminus \{3\}$, the change in the adjustment is not in the correct direction for agent j. Hence, we require an ordering where agent j does not receive h_3 before the swap but does receive it afterward. If $k \in \{1,2\}$, consider an ordering beginning with $k43k'j$, where $k' = 3-k$; if $k \notin \{1,2\}$, consider an ordering starting with $45123j$.

    - When the swapping agent is $i \in [m-2] \setminus \{3\}$: here, we consider only the swap where agent i exchanges h_3 with the house immediately above it, and only when that house is h_i, since in any other swap of agent i involving h_3, the swap does not change the adjustment. In this case, the change is not in the correct direction for agent m, so we must find an ordering where agent m receives h_3 before the swap but not afterward. When $i \neq 1$, an ordering beginning with $13ij$, followed by the other agents in $[m-2]$ in some order, and finally agent m, satisfies this requirement. When $i = 1$, a similar ordering with initial segment $231j$ serves the same purpose.

- When h_4 is involved in the swap, we proceed in a manner analogous to the case of h_3.

    - When the swapping agent is $i = 4$:

        - If h_4 is swapped upward with b, the adjustment for agent j is not in the correct direction. In this case, an ordering beginning with $54321j$ is one in which agent j receives h_4 before the swap but not afterward.

        - If h_4 is swapped downward with some h_k for $k \in [m-2] \setminus \{4\}$, the adjustment for agent m is not in the correct direction. When $k \neq 1$, an ordering beginning with $1kj$, followed by the other agents in $[m-2]$, and then agent m, is one in which agent m receives h_4 only after the swap. When $k = 1$, we consider a similar ordering beginning with $214j$.

    - When the swapping agent is $i \in [m-2] \setminus \{4\}$: we consider only the case in which h_4 is swapped upward with h_i. In this case, the change is not in the correct direction for agent j. Let $i', i'' \in [m-2] \setminus \{1,2,3,i\}$ be two distinct agents (such agents exist since $m \geq 8$). An ordering beginning with $i'i''$, followed by the agents in $\{1,2,3,i\}$ in some order (note that i may belong to $\{1,2,3\}$), and then agent j, is one in which agent j receives h_4 only before the swap.

Thus, g satisfies CM, and consequently f satisfies all the axioms stated in the proposition.

Finally, to show that $f \neq f_{\text{RSD}}$, we apply an argument similar to that used in Proposition 5.3. Consider a profile $\mathbf{Q} \in \mathsf{P}$ such that agent j's top choices are $h_1 Q_j h_2 Q_j h_3$; for every $i \in N \setminus \{j\}$, the top choices are $a Q_i b$; and for every $i \in N \setminus \{3, j\}$, the house h_3 is not among agent i's top five choices. Applying similar arguments to this profile establishes the desired distinction between f and f_{RSD}. ■

B Exhaustive case analysis for $n = m = 4$

Here, we exhaustively verify that for every profile, the assignment matrix induced by any mechanism satisfying ExPE, ETE, and SP (hereafter, *the axioms*) is uniquely determined. Let $H \coloneqq \{a, b, c, d\}$. We use the shorthand $xyzw$ to denote $xP_i^+ yP_i^+ zP_i^+ w$ when the ranking P_i is clear from the context. A preference profile is represented by a 4×4 table, where the i-th column specifies agent i's ranking. We use x, y, z to indicate multiple possible rankings or profiles, where x, y, z may be replaced by some houses such that the resulting sequence constitutes a valid ranking. For example, when we write $abxy$, this may refer to either $abcd$ or $abdc$. Throughout the verification of each profile, the symbols x, y, z are fixed.

For profiles that are not supported, we have already seen in Section 4.2 how their assignment matrices are determined by the induction hypothesis. For supported profiles, we may also apply the induction hypothesis. However, note that the lexicographic order in which we examine the profiles does not coincide with the order of induction. Therefore, whenever we invoke the induction hypothesis, we ensure that it applies only to profiles with a smaller disagreement parameter.

Remark 15. When determining the assignment matrix of a supported profile, we may also use unsupported agent rankings from profiles with the same disagreement parameter, since their corresponding column entries were determined solely based on profiles with smaller disagreement parameters.

To aid readability, we provide brief explanations after each profile table describing how its entries are determined. Since many of these arguments are similar or repetitive, we will give full explanations only for the first few instances of each argument type. Whenever an argument reappears later, we will omit the immediate explanation; if it is not immediate but still repetitive, we will summarize it in a remark and refer to that remark beside the relevant table.

For clarity, we will color each entry of the assignment matrix according to the reason for which it is determined:

- Red - determined by efficiency (the agent's probability of receiving that house is zero).
- Blue - determined by ETA.
- Green - determined by SP using another already-determined profile (by induction or otherwise).
- Purple - determined by complementing that agent's probabilities to one.
- Orange - determined by complementing that house's probabilities to one.
- Brown - determined by reasoning stated in a remark and referred to beside the relevant profile table.

Entries that will be determined in subsequent steps are underlined, and the corresponding entries to be determined in later profiles are **bolded**.

We denote by (i, h) the probability that a mechanism satisfying the axioms assigns house h to agent i in a given profile. Finally, when we say that *a profile is determined*, we mean that the assignment matrix induced by any mechanism satisfying the axioms is uniquely determined for that profile.

B.1 Profiles with two agents sharing the same ranking

We begin with the preference profiles in which two agents share the same ranking. Since the axioms are invariant under renamings of both agents and houses, we may, without loss of generality, assume that agents 1 and 2 share the same ranking, $abcd$, and that the ranking of agent 4 does not precede that of agent 3 in the lexicographic order.

Remark 16. Whenever agent 3 also ranks a first while agent 4 does not, agent 4 is not supported, because of the adjacent pair consisting of a and the house immediately above it. Hence, in such profiles the probabilities of agent 4 are determined.

Remark 17. Whenever agent 3's ranking is obtained from the ranking $abcd$ by an adjacent swap, the probabilities associated with the two houses that are not involved in that swap are determined. For example, when agent 3 ranks $abdc$, the probabilities associated with houses a and b are determined. This is because the only pair of houses for which the agents might not exhibit near-unanimous agreement is $\{c, d\}$. Thus, agents 1, 2, and 3 can swap $\{c, d\}$ in their rankings to reach a profile determined by Lemma 2.4 (hereafter, *the lemma*). By SP, such a swap does not affect their probabilities of receiving a or b. The probabilities of these houses for agent 4 are then determined by house complementarity. The same argument applies when agent 3 ranks $acbd$ or $bacd$: in the former case, the probabilities associated with a and d are determined, and in the latter, the probabilities associated with c and d are determined.

Case 1. When agent 3 also ranks $abcd$, the profile is determined by the lemma, since there are $n - 1 = 3$ agents with the same ranking. In particular, the agents exhibit near-unanimous agreement on the relative ranking of every pair of houses, and the profile is therefore determined.

Case 2. When agent 3 ranks $abdc$, note that the lemma determines every profile $\mathbf{P}$ satisfying cP_4d, because in such profiles there is no pair of houses on which the agents fail to exhibit near-unanimous agreement, and those profiles are therefore determined. We now go through the remaining possible rankings for P_4.

Case i.

a	a	a	a
b	b	b	b
c	c	d	d
$\underline{d}$	d	c	c

(17)

Since there are two pairs of agents with identical rankings, by ETE and complementing to one, it suffices to determine $(1, d)$. By SP, it suffices to determine $(1, d)$ in the following profile where agent 1 changes his ranking from $abcd$ to $cbad$.[3]

c	a	a	a
b	b	b	b
a	c	d	d
$\boldsymbol{d}$	$\underline{d}$	c	c

Here, $(1, b)$, $(1, a)$, $(3, c)$, and $(4, c)$ are determined by efficiency,[4] and consequently the remaining probabilities for houses a and b are determined by ETA. Next, $(3, d)$ and $(4, d)$ are determined by the agent's complement. By house complementarity, to determine $(1, d)$ it suffices to determine $(2, d)$; by SP, this in turn reduces to determining $(2, d)$ in the following profile.

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[3] The ranking of the agent can be modified through a sequence of adjacent swaps that does not involve d. By SP, the agent's probability of receiving d remains unchanged after each such swap.

c	c	a	a
b	b	$\underline{b}$	b
a	a	d	d
d	$\boldsymbol{d}$	c	c

$(1,a)$, $(2,a)$, $(3,c)$, and $(4,c)$ are determined by efficiency,[5] and consequently the remaining probabilities for a are determined by ETA. Since there are two pairs of agents with identical rankings, it suffices to determine $(3,b)$, and it suffices to do so in the following profile.

c	c	a	a
b	b	$\boldsymbol{b}$	b
a	a	c	d
d	d	d	c

$(1,a)$, $(2,a)$, and $(3,c)$ are determined by efficiency, and consequently the remaining probabilities for a are determined by ETA. Note that $(3,d)$ is determined by SP and by considering the profile where agent 3 changes his ranking to $cbad$ (i.e., matches the rankings of agents 1 and 2). That profile is determined by the lemma because there are three agents with the same ranking. Then, $(3,b)$ is determined by agent's complement, as required.

Case ii.

a	a	a	a
b	b	b	d
c	c	d	x
$\underline{d}$	d	c	y

(17)

First, note that agent 4 is not supported because we can see that all other agents have the opposite preference with respect to the adjacent pair containing b and the house immediately above it in agent 4's ranking (either c or d). Since agents 1 and 2 share the same ranking, it suffices to determine $(1,d)$ in the following profile.

c	a	a	a
b	b	b	d
a	c	d	x
$\boldsymbol{d}$	$\underline{d}$	c	y

$(1,b)$, $(1,a)$, $(3,c)$, $(4,b)$, and $(4,c)$ are determined by efficiency,[6] and consequently ETA determines a and b (the remaining probabilities for those houses), and $(3,d)$ and $(4,d)$ are determined by agent's complement. It then suffices to determine $(2,d)$ in the following profile.

c	c	a	a
b	b	$\underline{b}$	d
a	a	d	x
d	$\boldsymbol{d}$	c	y

$(1,a)$, $(2,a)$, $(3,c)$, $(4,b)$, and $(4,c)$ are determined by efficiency, and consequently a is determined by ETA, and $(4,d)$ is determined by agent's complement. Since agents 1 and 2 share the same ranking, it suffices to determine $(3,b)$ in the following profile.

c	c	a	a
b	b	$\boldsymbol{b}$	d
a	a	c	x
d	d	d	y

$(1,a)$, $(2,a)$, and $(3,c)$ are determined by efficiency, and consequently a is determined by ETA. $(3,d)$ is determined by the lemma (we can switch to the profile where agent 3 matches the ranking of agents 1 and 2), and then $(3,b)$ is determined by agent's complement, as required.

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[4] $(1,b)$ and $(1,a)$ are determined because agent 1 is the only agent preferring c over b and a. An assignment in which he receives b or a while another agent receives c would be inefficient, since swapping houses would strictly benefit both. $(3,c)$ and $(4,c)$ are determined because agents 3 and 4 rank c last, implying that they can receive it only if they are last in the ordering. Since agent 1 ranks c first, no agent following him in the ordering can receive it (if the house is available at his turn, he will take it).

[5] Each agent $i \in \{1,2\}$ ranks a second to last, implying that he receives it only when he is last or second to last, and in particular either agent 3 or agent 4 comes before agent i, while both agents 3 and 4 rank a first, which implies that any agent that comes after one of them in the ordering would not receive a.

[6] Note that agent 4 cannot receive b or c because for that to happen, some other agent should receive d beforehand, and the only agent other than 4 which can receive d when he is not the last in the ordering is agent 3 (because he does not rank d last). For that to happen, agents 1 and 2 should come before agent 3 and take the houses that he prefers over d, that is, a and b, but because of their preferences, they will receive a and c instead. Furthermore, agent 1 cannot receive b because for that to happen, another agent must receive c beforehand, but the only other agent who might receive c is agent 2, and he prefers b, so he would not take c if b were available when it is his turn.

Case iii.

a	a	a	b
b	b	b	x
c	c	d	y
d	d	$\underline{c}$	z

(16, 17)

It suffices to determine $(3, c)$ in the following profile.

a	a	d	b
b	b	a	x
c	c	b	y
d	d	**c**	z

$(1, d)$, $(2, d)$, $(3, a)$, $(3, b)$, and $(4, a)$ are determined by efficiency, and consequently a is determined by ETA. $(4, b)$ is determined by considering the profile where agent 4 ranks $bacd$ and applying the lemma. Then, $(1, b)$ and $(2, b)$ are determined by house complementarity, and then $(1, c)$ and $(2, c)$ are determined by agent complementarity. It then suffices to determine $(4, c)$ in the following profile.[7]

a	a	d	a
b	b	a	b
c	c	b	d
d	d	c	**c**

$(1, d)$, $(2, d)$, $(3, a)$, and $(3, b)$ are determined by efficiency, and consequently a and b are determined by ETA. Then, $(1, c)$ and $(2, c)$ are determined by agent's complement. To determine $(3, c)$, we consider the profile obtained by changing agent 3's ranking to $abdc$. Note that this new profile has a disagreement parameter smaller than that of the profiles we wish to solve, and is therefore determined under the induction hypothesis. $(4, c)$ is then determined by house complementarity, as required.

Case iv.

a	a	a	d
b	b	b	x
c	c	d	y
d	d	c	z

(16)

In such profiles, agents 1 and 2 are also not supported because of the pair $\{c, d\}$, and consequently those profiles are not supported.

Case 3. When agent 3 ranks $acbd$, note that all profiles where bP_4c are determined by the lemma, and therefore we only need to consider the remaining possible rankings for P_4.

Case i.

a	a	a	a
$\underline{b}$	b	c	c
c	c	b	b
d	d	d	d

(17)

It suffices to determine $(1, b)$ in the following profile.

a	a	a	a
b	b	c	c
d	c	b	b
c	d	d	d

$(1, c)$ is determined by efficiency, and a is determined by ETA. Note that since $\{b, c\}$ is the only pair without near-unanimous agreement among the agents, (i, d) is determined for $i = 2, 3, 4$, because each of those agents can swap $\{b, c\}$ and obtain a profile that is determined by the lemma. Then, $(1, b)$ is determined by complementarity, as required.

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[7] If $z = c$, the conclusion follows directly from SP. Otherwise, since we assume dP_4c, we must have $x = d$. In that case, $(4, a)$ is determined in both profiles with $x = d$, and therefore, by Remark 4, it suffices to determine $(4, c)$ at the profile where aP_4c.

Case ii.

a	a	a	a
$\underline{b}$	b	c	c
c	c	b	d
d	d	d	b

(17)

Agent 4 is not supported because of the pair $\{d, b\}$. It suffices to determine $(1, b)$ in the following profile.

a	a	a	a
$\boldsymbol{b}$	b	c	c
$\underline{d}$	c	b	d
c	d	d	b

$(1, c)$ is determined by efficiency, and a is determined by ETA. It suffices to determine $(1, d)$ in the following profile.

b	a	a	a
a	b	c	c
$\boldsymbol{d}$	c	b	$\underline{d}$
c	d	d	b

(i, d) for $i = 2, 3$ is determined because each of these agents can swap $\{b, c\}$ and obtain a profile that is determined by the lemma. It then suffices to determine $(4, d)$.

b	a	a	c
a	$\underline{b}$	$\underline{c}$	a
d	c	b	$\boldsymbol{d}$
c	d	d	b

$(1, a)$, $(1, c)$, $(4, a)$, and $(4, b)$ are determined by efficiency, and consequently a is determined by ETA. $(2, d)$ is determined since agent 2 can swap $\{b, c\}$ to obtain a profile determined by the lemma. It then suffices to determine $(2, b)$ and $(3, c)$.

b	a	a	c
a	$\boldsymbol{b}$	c	a
d	$\underline{d}$	b	d
c	c	d	b

  We can use this profile to determine $(2, b)$. $(1, a)$, $(2, c)$, and $(4, a)$ are determined by efficiency, and consequently a is determined by ETA. It suffices to determine $(2, d)$ in the following profile.

b	b	$\underline{a}$	c
a	a	c	a
d	$\boldsymbol{d}$	b	d
c	c	d	b

    $(2, c)$, $(3, b)$, $(4, a)$, and $(4, b)$ are determined by efficiency, and consequently b is determined by ETA. It then suffices to determine $(3, a)$ in the following profile (note that agents 1 and 2 share the same ranking).

b	b	$\boldsymbol{a}$	c
a	a	b	a
d	d	d	d
c	c	c	b

    Efficiency and ETA determine b, c, and d, and consequently the profile is determined.

b	a	a	c
a	b	$\boldsymbol{c}$	a
d	c	$\underline{d}$	d
c	d	b	b

  We can use this profile to determine $(3, c)$. $(1, a)$, $(3, b)$, and $(4, a)$ are determined by efficiency, and consequently a is determined by ETA. It then suffices to determine $(3, d)$ in the following profile.

b	$\underline{a}$	c	c
a	b	a	a
d	c	$\boldsymbol{d}$	d
c	d	b	b

    $(1, a)$, $(1, c)$, $(2, c)$, and $(3, b)$ are determined by efficiency, and consequently c is determined by ETA. It then suffices to determine $(2, a)$ in the following profile (note that agents 3 and 4 share the same ranking).

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b	$\boldsymbol{a}$	c	c
a	c	a	a
d	d	d	d
c	b	b	b

Efficiency and ETA determine b, c, and d, and consequently the profile is determined.

Case iii.

a	a	a	a
b	b	c	d
c	c	b	c
d	d	d	b

This profile is not supported because the pair $\{b, c\}$ is not under near-unanimous agreement among the agents, and both agents 3 and 4 cannot receive b.

Case iv.

a	a	a	c
$\underline{b}$	b	c	a
c	c	b	b
d	d	d	d

(16, 17)

It suffices to determine $(1, b)$ in the following profile.

a	a	a	c
$\boldsymbol{b}$	b	c	a
d	c	b	b
c	d	d	d

$(1, c)$ and $(4, a)$ are determined by efficiency, and consequently a is determined by ETA. (i, d) for $i = 2, 3, 4$ is determined by the lemma: agents 2 and 3 can swap $\{b, c\}$, and agent 4 can change his ranking to $abcd$; in both cases, the resulting profile is determined by the lemma. $(1, b)$ is then determined by complementarity, as required.

Case v.

a	a	a	c
$\underline{b}$	b	c	a
c	c	b	d
d	d	d	b

(16, 17)

It suffices to determine $(1, b)$ in the following profile.

a	a	a	c
$\boldsymbol{b}$	b	c	a
d	c	b	d
c	d	d	b

$(1, c)$, $(4, a)$, and $(4, b)$ are determined by efficiency, and consequently a is determined by ETA. (i, d) for $i = 2, 3$ is determined by the lemma: each of these agents can swap $\{b, c\}$, yielding a profile covered by the lemma. $(4, c)$ is determined by considering the profile where agent 4 swaps $\{d, b\}$; the disagreement parameter of that profile equals that of the original one,[8] and in that profile agent 4 is unsupported because of the pair $\{c, a\}$, so the desired result follows by Remark 15. $(1, b)$ is then determined by complementarity, as required.

Case vi.

a	a	a	c
b	b	c	b
c	c	b	x
d	d	d	y

We use similar arguments to those applied in *Case* iv. When agent 4 does not rank d last, we make a slight adjustment: in the second step, to determine $(4, d)$, we may use

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[8] This holds because the addition from swapping the pair $\{c, d\}$ in agent 1's ranking in the first transition is cancelled by the swap of the pair $\{d, b\}$ in agent 4's ranking in the second transition.

the lemma on the profile obtained when agent 4 swaps $\{b, c\}$ (that is, the profile where he ranks $bcda$).

Case vii.

a	a	a	c
$\underline{b}$	b	c	d
c	c	b	x
d	d	d	y

(16, 17)

It suffices to determine $(1, b)$ in the following profile.

a	a	a	c
$\boldsymbol{b}$	b	c	d
d	c	b	x
c	d	d	y

$(1, c)$, $(4, a)$, and $(4, b)$ are determined by efficiency, and consequently a is determined by ETA. (i, d) for $i = 2, 3$ is determined by the lemma: for each of these agents, swapping $\{b, c\}$ yields a profile covered by the lemma. $(4, c)$ is determined by considering the profile where agent 4 swaps $\{d, x\}$; the disagreement parameter of that profile equals that of the original one,[9] and in that profile agent 4 is unsupported because of the adjacent pair containing a and the house ranked immediately above it. $(1, b)$ is then determined by complementarity, as required.

Case viii.

a	a	a	d
b	b	c	x
c	c	b	y
d	d	d	z

(16, 17)

$(3, b)$ is determined by efficiency, and consequently the profile is determined.

Case 4. When agent 3 ranks $acdb$, all such profiles are not supported since both agents 3 and 4 are not supported. Agent 3 is not supported because of the pair $\{d, b\}$. For agent 4, Remark 16 shows that he is not supported whenever he does not rank a first, and when he does, he is still not supported because of the pair containing b and the house immediately above it (note that we only consider rankings for agent 4 that do not precede $acdb$ in lexicographic order).

Case 5. When agent 3 ranks $adbc$, all such profiles are not supported as well. If agent 4 ranks $adxy$, then agents 1 and 2 are not supported because of the pair $\{c, d\}$, which suffices for the case where he ranks $adbc$, since in that case agents 3 and 4 share the same ranking.[10] When agent 4 ranks $adcb$, he is not supported because of the pair $\{c, b\}$. In the remaining cases, agent 4 is not supported by Remark 16. When he ranks d first, agents 1 and 2 are not supported because of the pair $\{c, d\}$; when he ranks either b or c first, agent 3 is not supported because of the pair $\{d, b\}$.[11]

Case 6. When agent 3 ranks $adcb$, all such profiles are not supported as well. Agent 3 is not

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[9] This holds because the addition from swapping the pair $\{c, d\}$ in agent 1's ranking in the first transition is cancelled by the swap of the pair $\{d, x\}$ in agent 4's ranking in the second transition.

[10] Note also that this profile can be obtained from the one where agents 3 and 4 rank $acdb$ after suitable renamings of the agents and houses.

[11] Note that in this case, agent 3 cannot receive b, since another agent must receive d beforehand, and because agents 1 and 2 rank d last, that agent must be agent 4. However, he would not receive d unless agents 1 and 2 received his first preference earlier. If that preference is b, agent 3 would not receive it afterward; if it is c, then agents 1 and 2 cannot receive it when they are among the first two agents in the ordering, as it is only their third choice.

supported because of the pair $\{c, b\}$. Agent 4 is not supported either: if he shares the same ranking as agent 3, then it is because of the pair $\{c, b\}$; otherwise, he does not rank a first, and hence is not supported by Remark 16.

We now move to the cases where agent 3 does not rank a first, most of which are resolved by the following remark.

Remark 18. It suffices to check the cases where agent 3 ranks a second and where agents 3 and 4 have the same top choice. Indeed, whenever the top choices of agents 3 and 4 differ, neither of them would receive a, and both would be unsupported because of the pair containing a and the house ranked immediately above it in their orderings. Moreover, even when their top choice is the same, they could receive a only when they rank it second. Since we assume that the ranking of agent 4 does not precede that of agent 3 in the lexicographic order, it is impossible that agent 4 ranks a second while agent 3 does not.

We now verify the cases where agent 3 ranks a second and agent 4's top choice is the same as agent 3's.

Case 7. When agent 3 ranks $bacd$.

Case i.

$\underline{a}$	a	b	b
b	b	a	a
c	c	c	c
d	d	d	d

(17)

It suffices to determine $(1, a)$ in the following profile.

$\boldsymbol{a}$	a	b	b
c	b	a	a
b	c	c	c
d	d	d	d

$(1, b)$ is determined by efficiency, d is determined by ETA, and (i, c) for $i = 2, 3, 4$ is determined by considering the profile where agent i swaps $\{a, b\}$, which is determined by the lemma. $(1, a)$ is then determined by complementarity, as required.

Case ii.

$\underline{a}$	a	b	b
b	b	a	a
c	c	c	d
d	d	d	c

(17)

Agent 4 is not supported because of the pair $\{d, c\}$. It then suffices to determine $(1, a)$ in the following profile.

$\boldsymbol{a}$	a	b	b
c	b	a	a
b	c	c	d
d	d	d	c

$(1, b)$ and $(4, c)$ are determined by efficiency. $(1, d)$ is determined by considering the profile where agent 1 ranks $bacd$ and by the lemma, and similarly (i, c) for $i = 2, 3$ are determined by considering the profiles where each agent i swaps $\{a, b\}$. $(1, a)$ is then determined by complementarity, as required.

Case iii.

$\underline{a}$	a	b	b
b	b	a	c
c	c	c	x
d	d	d	y

(17)

Agent 4 is not supported because of the pair containing a and the house immediately above it. It then suffices to determine $(1,a)$ in the following profile.

$\boldsymbol{a}$	a	b	b
d	b	a	c
b	c	c	x
c	d	d	y

$(1,b)$ and $(1,c)$ are determined by efficiency. Then, (i,d) for $i=2,3$ is determined by considering the profile where agent i swaps $\{a,b\}$ and by the lemma. Since $(4,a)$ is determined by efficiency in both profiles, Remark 4 implies that it suffices to determine $(4,d)$ at the profile where aP_4d. In that profile, $(4,d)$ is determined by considering the profile where agent 4 changes his ranking to $abcd$ and by the lemma. $(1,a)$ is then determined (in both profiles) by complementarity, as required.

Case iv.

a	a	b	b
b	b	a	d
c	c	c	x
d	d	d	y

We use similar arguments to those applied in *Case* ii.[12]

Case 8. When agent 3 ranks $badc$, all such profiles are not supported.

- When agent 4 ranks either $badc$ or $bdxy$, agents 1, 2, and 3 are not supported because of the pairs $\{a,b\}$ and $\{c,d\}$.
- Otherwise, agent 4 ranks $bcxy$. In that case, agent 3 is not supported because of the pair $\{d,c\}$, and agent 4 is not supported because of the pair containing a and the house ranked immediately above it.

Case 9. Otherwise, let $h \in \{c,d\}$ denote the top choice of agents 3 and 4. Note that agents 1 and 2 are not supported because of the pair containing h and the house ranked immediately above it.

- When agent 4 does not rank a second, he is not supported because of the pair containing a and the house ranked immediately above it.
- When agent 4 does rank a second, if he has the same ranking as agent 3, then it suffices that only agents 1 and 2 are not supported for the profile to be determined. Otherwise, let h' and h'' denote the two houses in $H \setminus \{a,h\}$. Note that either agent 3 or agent 4 is not supported because of the pair $\{h',h''\}$.[13]

B.2 Profiles with all agents holding distinct rankings

In this part we examine the profiles in which all four agents hold different rankings. Our task is to determine whether such a profile is supported and to show that, whenever a profile is supported, the axioms uniquely determine its corresponding assignment matrix.

Recall that a profile is supported if it contains at least two supported agents with different rankings. Moreover, the axioms are invariant under renamings of both agents and houses. Since in the present setting all four agents have distinct rankings, the agents play symmetric roles. Taken together, these facts imply that we may assume, without loss of generality, that two supported

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[12] This time, agent 4 is not supported because of the pair containing a and the house ranked immediately above it.

[13] Exactly one of them orders $\{h',h''\}$ in the same way as agents 1 and 2, so the other must be unsupported.

agents are indexed as agents 1 and 2, that the houses are renamed so that agent 1's ranking is $abcd$, and that the ranking of agent 3 precedes that of agent 4 in the lexicographic order. Any supported profile with four distinct rankings can be brought into this normalized form by appropriate renamings of agents and houses.

The assumption that agents 1 and 2 are supported imposes a system of constraints on the rankings of agents 3 and 4. For each agent $i \in \{1,2\}$ and each adjacent pair xP_i^+y in his ranking, at least two other agents must agree with his comparison, except that at most one such requirement may be relaxed by one, provided that both houses in that pair are efficient for that agent. Once the rankings of agents 1 and 2 are fixed, these conditions determine certain comparisons that must be satisfied by agents 3 and 4. For example, if agent 2 ranks $adbc$, then incorporating his preferences into the constraints for agent 1 yields the following remaining requirements for agents 3 and 4: one of them must prefer a to b, one must prefer b to c, and both must prefer c to d. As before, at most one of these requirements may be relaxed by one when both houses in the corresponding pair are efficient for agent 1.

We now formalize the notation used to describe these constraints. Fix two rankings $P_1, P_2 \in \mathcal{R}$, a supported agent $i \in \{1,2\}$, and an adjacent pair xP_i^+y in his ranking. Define $\widetilde{R}^i_{xy} \in \{1,2\}$ to be the number of agents in $\{3,4\}$ who are required to prefer x to y so that agent i has at least two other agents agreeing with him on that comparison. Formally,

$$\widetilde{R}^i_{xy} := \begin{cases} 1 & \text{if } xP^+_{3-i}y, \\ 2 & \text{otherwise.} \end{cases}$$

For illustration, if agent 2 ranks $adbc$, then the constraints from agent 1 are $\widetilde{R}^1_{ab} = 1$, $\widetilde{R}^1_{bc} = 1$, and $\widetilde{R}^1_{cd} = 2$, and the constraints from agent 2 are $\widetilde{R}^2_{ad} = 1$, $\widetilde{R}^2_{db} = 2$, and $\widetilde{R}^2_{bc} = 1$.

For each supported agent $i \in \{1,2\}$, at most one of the constraints $\widetilde{R}^i_{xy}$ may be relaxed by one, and this is allowed only when both houses in that pair are efficient for that agent. We denote the chosen relaxation pair by $L_i \in (H \times H) \cup \{\varnothing\}$, where $L_i = \varnothing$ means that no relaxation is applied for agent i. The relaxed constraints are

$$R^i_{xy} := \begin{cases} \widetilde{R}^i_{xy} - 1 & \text{if } L_i = (x,y), \\ \widetilde{R}^i_{xy} & \text{otherwise.} \end{cases}$$

Finally, if (x,y) is an adjacent pair with x immediately above y in either P_1 or P_2, then the requirement imposed on agents 3 and 4 after applying L_1 and L_2 is

$$R_{xy} := \max_{i \in \{1,2\}} \left\{ R^i_{xy} \right\}.$$

Here, we adopt the convention that $R^i_{xy} := 0$ whenever $\{x,y\}$ is not an adjacent pair in P_i with xP_i^+y. We record some remarks that will be used repeatedly throughout the case analysis.

Remark 19. When determining the assignment matrix of a given profile, we may assume that any profile with the same disagreement parameter in which at least two agents share the same ranking is already determined, since all such profiles were exhaustively analyzed in the previous section.

Remark 20. Suppose agents 1 and 2 share an adjacent pair $\{x,y\}$, with agent 1 preferring x

and agent 2 preferring y. Then $\widetilde{R}^1_{xy} = 2$ and $\widetilde{R}^2_{yx} = 2$. Since agents 3 and 4 can satisfy both requirements R_{xy} and R_{yx} only when $R_{xy} + R_{yx} \leq 2$, supportedness of both agents 1 and 2 forces $L_1 = (x, y)$ and $L_2 = (y, x)$. Moreover, if more than one such conflicting pair exists, the assumption that both agents 1 and 2 are supported cannot hold.

Remark 21. Suppose agents 1 and 2 share an adjacent pair $\{x, y\}$ and both prefer x. Then $\widetilde{R}^1_{xy} = \widetilde{R}^2_{xy} = 1$. In this situation, relaxing the requirement at (x, y) for only one of the two agents is formally possible but always redundant. For instance, if $L_1 = (x, y)$ and $L_2 \neq (x, y)$, then $R_{xy} = 1$, so at least one of agents 3 and 4 prefers x to y; in such a profile, agent 1 already has two agents agreeing with him, so taking (x, y) as his relaxation pair removes no constraint and moreover adds the requirement that both x and y must be efficient for him. It follows that relaxing the requirement at (x, y) for only one agent never produces any new profiles to check; it only excludes profiles by adding unnecessary efficiency conditions. We may therefore assume without loss of generality that the relaxation at (x, y) is either applied for both agents or for neither of them.

Remark 22. For any three distinct houses $x, y, z \in H$, agents 3 and 4 can satisfy the three requirements R_{xy}, R_{yz}, and R_{zx} only when $R_{xy} + R_{yz} + R_{zx} \leq 4$. This is because each of agents 3 and 4 can satisfy at most two of the corresponding binary comparisons in his ranking. In particular, for any $i \in \{1, 2\}$, whenever $\widetilde{R}^i_{xy} = 2$, $\widetilde{R}^{3-i}_{yz} = 2$, and $\widetilde{R}^1_{zx} = \widetilde{R}^2_{zx} = 1$, one of these requirements must be relaxed through the choice of the relaxation pairs. Thus, we can assume that either $L_i = (x, y)$, or $L_{3-i} = (y, z)$, or $L_1 = L_2 = (z, x)$.

Remark 23. Whenever agent 2's ranking is obtained from $abcd$ by an adjacent swap, the probabilities associated with the two houses that are not involved in that swap for agents 1 and 2 are determined. For example, when agent 2 ranks $abdc$, for agents 1 and 2 to be supported, exactly one of the agents 3 and 4 must rank c above d. Consequently, whenever c and d are adjacent in the ranking of an agent, swapping them yields a profile with a strictly smaller disagreement parameter. Thus, the probabilities (i, h) for $i \in \{1, 2\}$ and $h \in \{a, b\}$ are determined by the induction hypothesis and SP. The same argument applies when agent 2 ranks $acbd$ and $h \in \{a, d\}$ and when agent 2 ranks $bacd$ and $h \in \{c, d\}$.

Remark 24. Note that whenever $R_{xd} > 0$ for some $x \in H \setminus \{d\}$, we may assume that agent 3 does not rank d first. This is because we assume that his ranking precedes that of agent 4 in the lexicographic order, so if he ranks d first, agent 4 ranks d first as well, and the constraint is not satisfied.

Definition B.1. A profile is called *degenerate* if each agent ranks a different house first.

Remark 25. Every degenerate profile is determined, because efficiency forces each agent to receive his top-ranked house.

Remark 26. Note that whenever agent 2 ranks b first and $R_{xc} > 0$ for some $x \in H \setminus \{c\}$, we may assume that agent 3 does not rank c first. This is because we assume that his ranking precedes that of agent 4 in the lexicographic order, so if he ranks c first, in order to satisfy the constraint agent 4 must rank d first, and we obtain a degenerate profile; hence it is determined by Remark 25.

Remark 27. Note that whenever agent 2 ranks b first and b is involved in the relaxation pair of agent 1 (that is, $L_1 = (a, b)$ or $L_1 = (b, c)$), we may assume that agent 3 does not rank b first.

This is because we assume that his ranking precedes that of agent 4 in the lexicographic order, so if he ranks b first, agent 4 cannot rank a first, and then agent 1 cannot receive b.

We now proceed to examine all possible profiles, grouped according to the ranking of agent 2.

Case 1. When agent 2 ranks $abdc$, $\{c,d\}$ is an adjacent pair in both P_1 and P_2, with agent 1 preferring c and agent 2 preferring d. By Remark 20, we must therefore have $L_1=(c,d)$ and $L_2=(d,c)$. The constraints imposed on the rankings of agents 3 and 4 are

$$R_{ab}=R_{bc}=R_{cd}=R_{bd}=R_{dc}=1.$$

Since $L_1=(c,d)$ and $L_2=(d,c)$, the houses c and d must both be efficient for agents 1 and 2, and consequently agents 3 and 4 cannot rank either c or d in the top position. Moreover, agent 3 cannot rank b first either: since its ranking precedes that of agent 4, this would force agent 4 to rank b first, but then the requirement $R_{ab}=1$ would not be satisfied. We now consider subcases according to the ranking of agent 3 (recall that we do not examine cases where two agents share the same ranking).

Case i. When agent 3 ranks $acbd$, this ranking satisfies aP_3b, cP_3d, and bP_3d. The constraints therefore require that agent 4's ranking satisfy bP_4c and dP_4c. After taking these restrictions into account as well, the remaining possibilities for P_4 are $adbc$, $badc$, and rankings of the form $bdxy$.

Case α.

a	a	a	a
b	b	c	d
c	d	b	b
$\underline{d}$	c	d	c

(23)

In this profile, agents 3 and 4 are not supported because of the pairs $\{c,b\}$ and $\{d,b\}$, respectively. It then suffices to determine $(1,d)$ in the following profile.

a	a	a	a
c	b	c	d
b	d	b	b
d	c	d	c

$(2,c)$, $(4,b)$, and $(4,c)$ are determined by efficiency, and consequently a and c are determined by ETA. $(2,b)$ is determined by considering the profile where agent 2 swaps $\{c,d\}$. That profile's disagreement parameter equals that of the original profile, and it is not supported.[14] $(1,d)$ is then determined by complementarity, since agents 1 and 3 share the same ranking.

Case β.

a	a	a	b
b	b	c	a
c	d	b	d
$\underline{d}$	c	d	c

(23)

In this profile, agents 3 and 4 are not supported because of the pairs $\{c,b\}$ and $\{b,a\}$, respectively. It then suffices to determine $(1,d)$ in the following profile.

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[14] In that profile, both agents 2 and 4 are not supported due to the pair $\{b,c\}$, and the remaining agents share the same ranking.

c	a	a	b
a	b	c	a
b	$\underline{d}$	b	$\underline{d}$
$\boldsymbol{d}$	c	$\underline{d}$	c

By house complementarity, it suffices to determine (i, d) for $i = 2, 3, 4$.

c	b	$\underline{a}$	b
a	a	c	a
b	$\boldsymbol{d}$	b	d
d	c	d	c

  We can use this profile to determine $(2, d)$. $(1, a)$, $(1, b)$, $(2, c)$, and $(3, b)$ are determined by efficiency, and consequently b is determined by ETA. It suffices to determine $(3, a)$ in the following profile (since agents 2 and 4 share the same ranking).

c	b	$\boldsymbol{a}$	b
a	a	b	a
b	d	d	d
d	c	c	c

    Efficiency and ETA determine b, c, and d, and consequently the profile is determined.

c	a	a	b
a	b	b	a
b	d	c	d
d	c	$\boldsymbol{d}$	c

  We can use this profile to determine $(3, d)$. $(1, a)$, $(1, b)$, $(2, c)$, $(4, a)$, and $(4, c)$ are determined by efficiency, and consequently a is determined by ETA. $(3, b)$ is determined by considering the profile where agent 3 swaps $\{c, d\}$ and applying the lemma. $(1, d)$ is determined by considering the profile where agent 1 swaps $\{c, a\}$. That profile's disagreement parameter equals that of the original profile, and agent 1 is not supported there.[15] By Remark 15, the entries of that agent are determined in the swapped profile, and therefore $(1, d)$ is determined in the current profile. $(3, d)$ is then determined by complementarity.

c	a	a	a
a	b	c	b
b	d	b	$\boldsymbol{d}$
d	c	d	c

  We can use this profile to determine $(4, d)$. $(1, d)$ is determined by considering the profile where agent 1 swaps $\{c, a\}$. That profile has a smaller disagreement parameter than the original one. $(3, d)$ is determined by considering the profile where agent 3 swaps $\{c, b\}$. That profile's disagreement parameter equals that of the original profile, and it is not supported.[16] Since agents 2 and 4 share the same ranking, $(4, d)$ is then determined.

Case γ.

a	a	a	b
b	b	c	d
c	d	b	x
d	$\underline{c}$	d	y

(23)

In this profile, agents 3 and 4 are not supported: agent 3 because of the pair $\{c, b\}$, and agent 4 because of the pair containing a and the house

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[15] Agent 1 is not supported in that profile because of the pair $\{c, b\}$.

[16] That profile is not supported because, for agent 1 we may consider $\{c, a\}$, and for agents 2 and 4 we may consider $\{d, c\}$.

immediately above it. It then suffices to determine $(2, c)$ in the following profile.

a	d	a	b
b	a	c	d
c	b	b	x
d	$\boldsymbol{c}$	d	y

$(1, d)$, $(2, a)$, $(2, b)$, $(3, b)$, $(3, d)$, and $(4, a)$ are determined by efficiency. $(4, b)$ is determined by considering the profile where agent 4 ranks $bacd$ and applying the lemma. Since $(4, a)$ is determined by efficiency in both profiles, Remark 4 implies that it suffices to determine $(4, c)$ at the profile where aP_4c. In that profile, $(4, c)$ is determined by considering the profile where agent 4 ranks $abdc$. That profile's disagreement parameter is at most that of the original profile, and it is not supported.[17] $(2, c)$ is then determined by complementarity.

Case ii. When agent 3 ranks $acdb$, the constraints require that bP_4dP_4c. Thus, the remaining possibilities for P_4 are of the form $bxyz$ with dP_4c.

Remark 28. In this case and the next two cases, agent 3 is not supported because of the pair containing b, and agent 4 is not supported because of the pair containing a, where in both instances we refer to the pair formed with the house immediately above them in their respective rankings.

a	a	a	b
b	b	c	x
c	d	d	y
$\underline{d}$	c	b	z

(23,28)

It suffices to determine $(1, d)$ in the following profile.

$\underline{c}$	a	a	$\underline{b}$
a	b	c	x
b	$\underline{d}$	d	y
$\boldsymbol{d}$	c	b	z

$(1, a)$, $(2, c)$, $(3, b)$, and $(4, a)$ are determined by efficiency, and consequently a is determined by ETA. It then suffices to determine $(1, c)$, $(2, d)$, and $(4, b)$.

$\boldsymbol{c}$	a	a	b
a	b	c	x
$\underline{d}$	d	d	y
b	c	b	z

  We can use this profile to determine $(1, c)$. $(1, a)$ and $(1, b)$ are determined by efficiency. It then suffices to determine $(1, d)$ in the following profile.

a	a	a	b
c	b	c	x
$\boldsymbol{d}$	d	d	y
b	c	b	z

    $(1, b)$, $(2, c)$, and $(4, a)$ are determined by efficiency. Since dP_4c, $(4, c)$ is also determined by efficiency, and consequently a and c are determined by ETA. $(1, d)$ is then determined by complementarity.

c	b	a	b
a	a	c	x
b	$\boldsymbol{d}$	d	y
d	c	b	z

  We can use this profile to determine $(2, d)$. $(1, a)$, $(1, b)$, $(2, c)$, and $(3, b)$ are determined by efficiency, and consequently b is determined by ETA. $(3, a)$ is determined by considering the profile where agent 3

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[17] In that profile, agents 1, 2, and 3 are not supported because of the pairs $\{c, d\}$, $\{d, a\}$, and $\{c, b\}$, respectively.

ranks $abdc$. That profile's disagreement parameter is at most that of the original profile, and agent 3 is not supported there.[18] Note that if $x = a$, then agents 2 and 4 share the same ranking, and otherwise, since dP_4c, $(4, a)$ is determined by efficiency. In either case, $(2, d)$ is then determined by complementarity.

c	a	a	$\boldsymbol{b}$
a	b	c	a
b	d	d	c
d	c	b	d

  We can use this profile to determine $(4, b)$. $(3, b)$ is determined by efficiency. $(1, b)$ is determined by considering the profile where agent 1 swaps $\{c, a\}$. That profile's disagreement parameter is at most that of the original profile, and it is not supported.[19] Similarly, $(2, b)$ is determined by considering the profile where agent 2 swaps $\{d, c\}$.[20] $(4, b)$ is then determined by complementarity.

Case iii. When agent 3 ranks $adbc$, the constraints require that cP_4d and bP_4d. Thus, the remaining possibilities for P_4 are $bacd$ and rankings of the form $bcxy$.

Case α.

a	a	a	b
b	b	d	a
$\underline{c}$	d	b	c
d	c	c	d

(23,28)

It suffices to determine $(1, c)$ in the following profile.

b	a	a	b
a	b	d	a
$\boldsymbol{c}$	d	b	c
d	$\underline{c}$	c	d

$(3, c)$ is determined by considering the profile where agent 3 swaps $\{d, b\}$. That profile's disagreement parameter is less than that of the original profile. It then suffices to determine $(2, c)$ in the following profile (since agents 1 and 4 share the same ranking).

b	d	a	b
a	b	d	a
c	a	b	c
d	$\boldsymbol{c}$	c	d

$(1, d)$, $(2, b)$, $(2, a)$, $(3, b)$, and $(4, d)$ are determined by efficiency. $(3, a)$ is determined by considering the profile where agent 3 ranks $abcd$. That profile is determined by the lemma. $(3, c)$ is determined by considering the profile where agent 3 ranks $badc$. That profile's disagreement parameter equals that of the original profile, and it is not supported.[21] $(2, c)$ is then determined by complementarity.

Case β.

a	a	a	b
b	b	d	c
c	d	b	x
$\underline{d}$	c	c	y

(23,28)

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[18] Agent 3 is not supported in that profile because of the pair $\{a, b\}$.

[19] In that profile, agents 2, 3, and 4 are not supported because of the pairs $\{d, c\}$, $\{d, b\}$, and $\{b, a\}$, respectively.

[20] In that profile, agents 1, 3, and 4 are not supported because of the pairs $\{c, a\}$, $\{d, b\}$, and $\{b, a\}$, respectively.

[21] In that profile, agents 1 and 4 are not supported because of the pair $\{c, d\}$, and agent 2 is not supported because of the pair $\{d, b\}$.

It suffices to determine $(1,d)$ in the following profile.

c	a	a	b
a	b	d	c
b	d	b	x
$\boldsymbol{d}$	c	c	y

$(1,a)$, $(1,b)$, $(2,c)$, $(3,b)$, $(3,c)$, and $(4,a)$ are determined by efficiency, and consequently a is determined by ETA. $(4,b)$ is determined by considering the profile where agent 4 ranks $badc$. That profile is determined by the lemma. $(4,d)$ is determined by considering the profile where agent 4 ranks $abcd$.[22] That profile's disagreement parameter is at most that of the original profile, and it is not supported.[23] $(1,d)$ is then determined by complementarity.

Case iv. When agent 3 ranks $adcb$, the constraints require that bP_4cP_4d. Thus, the remaining possibilities for P_4 are of the form $bxyz$ with cP_4d.

a	a	a	b
b	b	d	x
c	d	c	y
$\underline{d}$	c	b	z

(23,28)

It suffices to determine $(1,d)$ in the following profile.

a	a	a	b
c	b	d	x
b	d	c	y
$\boldsymbol{d}$	$\underline{c}$	b	z

$(1,b)$, $(3,b)$, and $(4,a)$ are determined by efficiency, and consequently a is determined by ETA. $(2,b)$, $(3,d)$, and $(4,d)$ are determined by considering, for each of these agents, a modified profile whose disagreement parameter is at most that of the original profile and which is either not supported or in which that agent is not supported.[24] It then suffices to determine $(2,c)$ in the following profile.

a	a	a	b
c	d	d	x
b	b	c	y
d	$\boldsymbol{c}$	b	z

$(1,b)$, $(1,d)$, $(3,b)$, $(4,a)$, and $(4,d)$ are determined by efficiency, and consequently a and d are determined by ETA. $(4,c)$ is determined by considering the profile where agent 4 ranks $abcd$.[25] That profile's disagreement parameter is less than that of the original profiles. $(2,c)$ is then determined by complementarity.

Case 2. When agent 2 ranks $acbd$, by Remark 20, we must have $L_1=(b,c)$ and $L_2=(c,b)$. The constraints imposed on the rankings of agents 3 and 4 are

$$R_{ab}=R_{bc}=R_{cd}=R_{ac}=R_{cb}=R_{bd}=1.$$

Case i. When agent 3 ranks $abdc$, the constraints require that cP_4d and cP_4b.

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[22] Note that $(4,a)$ is determined by efficiency in both profiles of that form. Hence, by Remark 4, we may use this transition to determine $(4,d)$.

[23] In that profile, agents 1, 2, and 3 are not supported because of the pairs $\{c,a\}$, $\{d,c\}$, and $\{d,b\}$, respectively.

[24] For $(2,b)$, we consider the profile where agent 2 swaps $\{d,c\}$. In that profile, agents 1 and 3 are not supported because of the pair $\{c,b\}$, and agent 4 is not supported because of the pair containing a and the house immediately above it. For $(3,d)$, we consider the profile where agent 3 swaps $\{c,b\}$. In that profile, agent 3 is not supported because of the pair $\{d,b\}$. For $(4,d)$, since cP_4d, by Remark 4 when $z=a$, we may consider the profile where agent 4 ranks $cabd$. In that profile, agent 4 is not supported because of the pair $\{c,a\}$.

[25] Since $(4,a)$ is determined by efficiency at all the relevant profiles, by Remark 4 we may assume aP_4c.

Thus, the remaining possibilities for P_4 are $acdb$ and rankings of the form $cxyz$.

Case α.

a	a	a	a
$\underline{b}$	c	b	c
c	b	d	d
d	d	c	b

(23)

In this profile, agents 3 and 4 are not supported because of the pairs $\{d, c\}$ and $\{d, b\}$, respectively. It then suffices to determine $(1, b)$ in the following profile.

a	a	a	a
$\boldsymbol{b}$	c	b	c
d	b	d	d
c	d	c	b

$(1, c)$, $(3, c)$, and $(4, b)$ are determined by efficiency, and consequently a and c are determined by ETA. $(2, d)$ is determined by considering the profile where agent 2 ranks $abcd$. That profile's disagreement parameter equals that of the original profile, and agents 1 and 3 share the same ranking in it, so that profile is determined by Remark 19.

Case β.

a	a	a	c
$\underline{b}$	c	b	x
c	b	d	y
d	d	c	z

(23)

In profiles of this form, agents 3 and 4 are not supported: agent 3 because of the pair $\{d, c\}$, and agent 4 because of the pair containing a and the house immediately above it. It then suffices to determine $(1, b)$ in the following profile.

a	a	a	c
$\boldsymbol{b}$	c	b	x
d	b	d	y
c	d	c	z

$(1, c)$ and $(4, a)$ are determined by efficiency, and consequently a is determined by ETA. $(2, d)$ is determined by the profile where agent 2 swaps $\{c, b\}$. That profile's disagreement parameter equals that of the original profile,[26] and agents 1 and 3 share the same ranking in it, so that profile is determined by Remark 19. $(4, d)$ is determined by considering a profile where agent 4 ranks either $acbd$ or $acdb$, while preserving his binary preference between b and d from the given profile.[27] That profile's disagreement parameter is less than that of the original profile.[28] Since agents 1 and 3 share the same ranking, $(1, b)$ is then determined by complementarity.

Case ii. When agent 3 ranks $acdb$, the constraints require that bP_4c and bP_4d. Thus, the remaining possibilities for P_4 are of the form $bxyz$.

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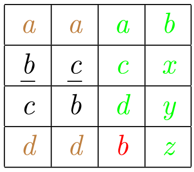
[26] This is true in all options, since agent 4 ranks c first, and therefore his binary preference over the pairs $\{c, d\}$ and $\{c, b\}$ does not change across them.

[27] Since agent 4 cannot receive a under any ranking of the form $cxyz$, we may assume aP_4d. This is justified as follows: if a and d are not adjacent in P_4, we may first swap $\{a, b\}$; by SP, this does not affect $(4, d)$. Once a and d become adjacent in P_4, swapping them leaves $(4, d)$ unchanged by Remark 4.

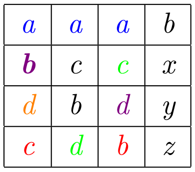
[28] This is because all the swaps considered in agent 4's ranking elevate a, and each such swap reduces the disagreement parameter by 3.



a	a	a	b
$\underline{b}$	$\underline{c}$	c	x
c	b	d	y
d	d	b	z

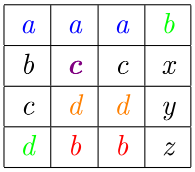
(23)

In profiles of this form, agents 3 and 4 are not supported: agent 3 because of the pair $\{d,b\}$, and agent 4 because of the pair containing a and the house immediately above it. It then suffices to determine either $(1,b)$ or $(2,c)$.

- 

a	a	a	b
$\boldsymbol{b}$	c	c	x
d	b	d	y
c	d	b	z

  When cP_4d, we use this profile to determine $(1,b)$. $(1,c)$, $(3,b)$, and $(4,a)$ are determined by efficiency, and consequently a is determined by ETA. $(2,d)$ is determined by considering the profile where agent 2 swaps $\{c,b\}$, and $(4,d)$ is determined by considering the profile where agent 4 ranks $acbd$.[29] Both profiles are determined by the lemma.[30] $(3,c)$ is determined by considering the profile where agent 3 swaps $\{d,b\}$. That profile's disagreement parameter equals that of the original profile,[31] and agents 2 and 3 share the same ranking in it, so that profile is determined by Remark 19. $(1,b)$ is then determined by complementarity.

- 

a	a	a	b
b	$\boldsymbol{c}$	c	x
c	d	d	y
d	b	b	z

  When dP_4c, we can use this profile to determine $(2,c)$. $(2,b)$, $(3,b)$, $(4,a)$, and $(4,c)$ are determined by efficiency, and consequently a is determined by ETA. $(1,d)$ is determined by considering the profile where agent 1 swaps $\{b,c\}$. That profile's disagreement parameter equals that of the original profile, and it is not supported.[32] $(4,b)$ is determined by considering the profile where agent 4 ranks $bacd$. The disagreement parameter of that profile is less than that of the original one.[33] Since agents 2 and 3 share the same ranking, $(2,c)$ is then determined by complementarity.

Case iii. When agent 3 ranks $adxy$, the constraints require that yP_4xP_4d. But then

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[29] Since agent 4 cannot receive a under any ranking of the form $bxyz$, we may assume aP_4d by Remark 4.

[30] In the profile where agent 2 swaps $\{c,b\}$, note that three agents prefer a to every other house (agents 1, 2, and 3), three agents prefer b to both c and d (agents 1, 2, and 4), and three agents prefer c to d (agents 2, 3, and, by assumption, agent 4).

[31] This is true in all relevant options, since agent 4's binary preference over the pairs $\{c,d\}$ and $\{d,b\}$ does not change across them.

[32] In that profile, agents 2 and 3 are not supported because of the pair $\{d,b\}$, and agent 4 is not supported because of the pair containing a and the house immediately above it. Moreover, the disagreement parameter equals that of the original profile in all options, since agent 4's binary preference over the pairs $\{b,d\}$ and $\{b,c\}$ does not change between them, as he ranks b first.

[33] In order to reach that ranking, agent 4 must swap the pair $\{c,d\}$, and that swap lowers the disagreement parameter by 3. Agent 4 may also perform additional swaps that elevate a, each of which decreases it even more. Thus this transition yields a net reduction of at least 3, whereas the earlier $\{b,d\}$ swap in agent 2's ranking raised it by only 1.

the agent $i \in \{1,2\}$ who ranks $axyd$ cannot receive y,[34] which contradicts the requirement imposed by that agent's relaxation pair.

Case iv. Otherwise, agent 3 does not rank a first. We first show that every profile satisfying the constraints in this case must be one in which agent 3 ranks b first and prefers a to c, and agent 4 ranks c first and prefers a to b. Since $R_{cd} = 1$, by Remark 24, we may assume that agent 3 does not rank d first. If agent 3 were to rank c first, then the requirement $R_{bc} = 1$ would imply that agent 4 cannot rank c first, so he must rank d first; but then agent 1 would be unable to receive c.[35] This contradicts the requirement imposed by the relaxation pair $L_1 = (b,c)$. Thus, agent 3 must rank b first. This, in turn, implies that the constraints require aP_4b and cP_4b. In particular, agent 4 cannot rank b first. He also cannot rank d first, since in that case agent 2 would be unable to receive b.[36] Thus, he must rank c first. This in turn implies that the constraints require aP_3c, so agent 3 must rank b first and prefer a to c, while agent 4 must rank c first and prefer a to b. We now consider all remaining possibilities collectively.

a	a	b	c
$\underline{b}$	c	x	$\widetilde{x}$
c	b	y	$\widetilde{y}$
d	d	z	$\widetilde{z}$

(23)

In profiles of this form, both agents 3 and 4 are not supported because of the pair containing a and the house immediately above it. It then suffices to determine $(1,b)$ in the following profile.

a	a	b	c
$\boldsymbol{b}$	c	x	$\widetilde{x}$
d	b	y	$\widetilde{y}$
c	d	z	$\widetilde{z}$

$(1,c)$, $(3,a)$, and $(4,a)$ are determined by efficiency, and consequently a is determined by ETA. It then suffices to determine (i,d) for $i = 2,3,4$. $(2,d)$ is determined by considering the profile where agent 2 swaps $\{c,b\}$. If cP_3d (i.e., the ranking of agent 3 is $bacd$), then that profile is determined by the lemma.[37] Otherwise, that profile's disagreement parameter equals that of the original profile, and it is not supported.[38] $(3,d)$ is determined as follows: if cP_3d, then it is determined by considering the profile where agent 3 ranks $acbd$. That profile's disagreement parameter is less than

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[34] Agent i cannot receive y because agent 3 is the only other agent who prefers x to y, so agent 3 must receive x first. For this to occur, either agent 4 or agent $3-i$ must receive d beforehand, yet both of them prefer y over d.

[35] Agent 1 cannot receive c because agent 4 is the only other agent who prefers b to c, so agent 4 must receive b first. For this to occur, either agent 2 or 3 must receive d beforehand, yet both of them prefer c over d.

[36] To see this, apply the argument used in the previous footnote, and exchange the roles of agents 1 and 2 and houses b and c.

[37] This is because agents 1, 2, and 4 prefer a to b; agents 1, 2, and 3 prefer a and b to both c and d; and agents 2, 3, and 4 prefer c to d.

[38] In this case, the swap of the pair $\{c,d\}$ in agent 1's ranking increases the disagreement parameter by only 1, so the subsequent swap of $\{c,b\}$ in agent 2's ranking offsets that increase entirely. In the resulting profile, agent 1 is not supported because of the pair $\{d,c\}$, and agents 3 and 4 are not supported because of the pair containing a and the house immediately above it.

that of the original profile.[39] Otherwise, $(3,c)$ is determined by efficiency, and $(3,b)$ is determined by considering the profile where agent 3 ranks $bacd$. That profile's disagreement parameter is at most that of the original profile, and agent 3 is not supported there.[40] $(3,d)$ is then determined by complementarity. Finally, $(4,d)$ is determined by considering the profile where agent 4 ranks either $acbd$ or $acdb$, while preserving his binary preference between b and d from the given profile.[41] That profile's disagreement parameter is at most that of the original profile, and it is not supported.[42]

Case 3. When agent 2 ranks $acdb$, the constraints from agent 1 are $\widetilde{R}^1_{ab}=1$, $\widetilde{R}^1_{bc}=2$, and $\widetilde{R}^1_{cd}=1$, and the constraints from agent 2 are $\widetilde{R}^2_{ac}=1$, $\widetilde{R}^2_{cd}=1$, and $\widetilde{R}^2_{db}=2$. By Remark 22 applied with $i=2$ and $(x,y,z)=(d,b,c)$, we must have either $L_2=(d,b)$, or $L_1=(b,c)$, or $L_1=L_2=(c,d)$.

Case i. When $L_2=(d,b)$, note that agents 3 and 4 cannot rank b first, because in that case agent 2 would be unable to receive it. We split according to the value of L_1.

If $L_1=\varnothing$ or $L_1=(a,b)$, then $R_{bc}=2$ and $R_{cd}=1$. These constraints imply that one of the agents $j\in\{3,4\}$ must satisfy bP_jcP_jd. Since agent 1 ranks $abcd$ and all four agents have distinct rankings, this forces agent j to rank b first, but then, as noted earlier, agent 2 would not be supported.

Otherwise, by Remark 21, $L_1=(b,c)$. Then, the constraints imposed on the rankings of agents 3 and 4 are

$$R_{ab}=R_{bc}=R_{cd}=R_{ac}=R_{db}=1.$$

Note that by Remark 24, we may assume that agent 3 does not rank d first.

- When agent 3 ranks $abdc$, the constraints require that cP_4dP_4b, but in that case agent 2 cannot receive b.[43]
- When agent 3 ranks $acbd$, the constraints require that dP_4bP_4c, but in

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[39] The transition from $bacd$ to $acbd$ in agent 3's ranking reduces the disagreement parameter by 4, whereas the swap of the pair $\{c,d\}$ in agent 1's ranking increased it by only 3.

[40] In this case, agent 3's transition reduces the parameter by at least the amount by which swapping the pair $\{c,d\}$ in agent 1's ranking increased it, since agent 3's transition swaps $\{c,d\}$ in the opposite direction, possibly after also swapping the pair $\{d,a\}$, which decreases the parameter even further. In the resulting profile, agent 3 is not supported because of the pair $\{b,a\}$.

[41] A similar argument to the one used in footnote 27 applies here.

[42] The swap of the pair $\{c,d\}$ in agent 1's ranking increases the disagreement parameter by at most 3, while the swap of the pair $\{c,a\}$ in agent 4's ranking decreases it by exactly 3 (since aP_3c). Any additional swaps that may occur in agent 4's transition elevate a, and these also decrease the parameter, as agents 1 and 2 rank a first. In the resulting profile, agent 1 is not supported because of the pair $\{d,c\}$, and agent 3 is not supported because of the pair containing a and the house immediately above it. Moreover, if dP_4b, then agent 4 is not supported because of that pair, and otherwise agents 2 and 4 share the same ranking.

[43] In that case, agent 2 cannot receive b because for that to occur, agent 4 would have to receive d beforehand (as he is the only other agent who ranks d over b). For agent 4 to receive d, either agent 1 or agent 3 would need to receive c first, yet both of them prefer b over c.

that case agent 1 cannot receive c.[44]

- When agent 3 ranks $adxy$, the constraints require that cP_4d. In this case, agent 4 cannot rank a first because agent 3's ranking precedes his, and as noted earlier he cannot rank b first either. Thus, since cP_4d, he must rank c first, but then agent 1 cannot receive c.[45]
- As noted earlier, when agent 3 ranks b first, agent 2 cannot receive it.
- Otherwise, agent 3 ranks c first. Then the constraints require that bP_4c, so agent 4 cannot rank c first. Since agent 3's ranking precedes agent 4's, the latter must rank d first. In that case, agent 1 cannot receive c.[44]

Case ii. When $L_1 = (b,c)$, it remains to consider the cases where $L_2 = \varnothing$ and $L_2 = (a,c)$. In these cases, we have $R_{db} = 2$, $R_{cd} = 1$, and $R_{bc} = 1$. These constraints imply that exactly one agent $i \in \{3,4\}$ must rank b over c, and that agent must rank d over b. Consequently, agent 1 cannot receive c.[46]

Case iii. When $L_1 = L_2 = (c,d)$, we have $R_{db} = R_{bc} = 2$. Since agents 3 and 4 have different rankings, at least one of them must rank d first. In that situation, agent 1 cannot receive d.

Case 4. When agent 2 ranks $adbc$, we may rename the houses so that his ranking becomes $abcd$. Under this renaming, agent 1's ranking becomes $acdb$. Swapping the names of agents 1 and 2 then reduces this case to the previous one.[47]

Case 5. When agent 2 ranks $adcb$, agents 1 and 2 disagree on both adjacent pairs $\{b,c\}$ and $\{c,d\}$, and by Remark 20, agents 3 and 4 cannot satisfy the constraints.

Case 6. When agent 2 ranks $bacd$, Remark 20 implies that $L_1 = (a,b)$ and $L_2 = (b,a)$. Under these conditions, the constraints become

$$R_{ab} = R_{bc} = R_{cd} = R_{ba} = R_{ac} = 1.$$

Note that by Remarks 24, 26, and 27, we may assume that agent 3 ranks a first.

Remark 29. In all subsequent subcases where agent 3 ranks a first, agent 4 must rank b first. Otherwise, agent 2 would be unable to receive a.

Case i. When agent 3 ranks $abdc$, the constraints require that cP_4d. Thus, by Remark 29, the remaining possibilities for agent 4's ranking are of the form $bcxy$ (because agent 2 ranks $bacd$).

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[44] This is because agent 4 would have to receive b beforehand, and for that to occur either agent 2 or agent 3 would need to receive d first, yet both prefer c over d.

[45] This is because agent 3 would need to prefer b over c and receive b, and for that to occur either agent 2 or agent 4 would need to receive d first, yet both prefer c over d.

[46] For that to occur, agent i would need to receive b beforehand, and for this to happen, one of the two other agents would need to receive d first, yet both prefer c over d.

[47] We may also swap the names of agents 3 and 4 so that the ranking of agent 3 precedes that of agent 4.

$\underline{a}$	b	a	b
b	a	b	c
c	c	d	x
d	d	c	y

(23)

In profiles of this form, agents 3 and 4 are not supported: agent 3 because of the pair $\{d,c\}$, and agent 4 because of the pair containing a and the house immediately above it. It then suffices to determine $(1,a)$ in the following profile.

$\boldsymbol{a}$	b	a	b
c	a	b	c
b	c	d	x
d	d	c	y

$(1,b)$, $(3,c)$, and $(4,a)$ are determined by efficiency. Since $\{a,b\}$ is the only pair on which the agents do not have near unanimous agreement, $(2,c)$, $(2,d)$, and $(3,d)$ are determined by considering the profiles where agent $i \in \{2,3\}$ swaps $\{a,b\}$ and applying the lemma. Likewise, $(1,d)$ is determined by considering the profile where agent 1 ranks $bacd$.[48] $(4,b)$ is determined by considering the profile where agent 4 ranks $bacd$. That profile's disagreement parameter is at most that of the original profile,[49] and agents 2 and 4 share the same ranking in it, so that profile is determined by Remark 19. Finally, $(1,a)$ is determined by complementarity.

Case ii. When agent 3 ranks $acxy$, Remark 29 implies that agent 4 must rank b first. His ranking cannot be $bacd$, since this is agent 2's ranking, and in this situation the constraints do not impose any additional restrictions on the remaining options.

$\underline{a}$	$\underline{b}$	a	b
b	a	c	$\tilde{x}$
c	c	x	$\tilde{y}$
d	d	y	$\tilde{z}$

(23)

In profiles of this form, agents 3 and 4 are not supported: agent 3 because of the pair containing b and the house immediately above it, and agent 4 because of the pair containing a and the house immediately above it, except in the ranking $badc$, where he is not supported because of the pair $\{d,c\}$. It then suffices to determine either $(1,a)$ or $(2,b)$.

$\boldsymbol{a}$	b	a	b
d	a	c	c
b	c	b	$\tilde{y}$
c	d	d	$\tilde{z}$

  When cP_4d and bP_3d, we use this profile to determine $(1,a)$.[50] $(1,b)$, $(1,c)$, and $(4,a)$ are determined by efficiency. (i,d) for $i=2,4$ is determined by considering the profile where agent i ranks $abcd$,[51] and $(3,d)$ is determined by considering the profile where agent 3 ranks $bacd$. All

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[48] Note that agents 2, 3, and 4 all prefer b to c. Therefore, agent 1 may swap $\{b,c\}$ without causing that pair to lose its near unanimous agreement among the agents.

[49] Swapping the pair $\{b,c\}$ in agent 1's ranking increased the disagreement parameter by 3. In agent 4's transition, swapping the pair $\{c,a\}$ decreases it by 3, and an additional swap of $\{a,d\}$, if it occurs, decreases it even further.

[50] Since agent 4's ranking is not $bacd$, he must rank c in the second position.

[51] If dP_4a, then by Remark 4 the transition to $abcd$ is justified.

these profiles are determined by the lemma.[52] $(1,a)$ is then determined by complementarity.

$\boldsymbol{a}$	b	a	b
c	a	c	c
d	$\underline{c}$	d	$\widetilde{y}$
b	$\underline{d}$	b	$\widetilde{z}$

- When cP_4d and dP_3b, we use this profile to determine $(1,a)$.[50] $(1,b)$, $(3,b)$, and $(4,a)$ are determined by efficiency, and consequently b is determined by ETA. It then suffices to determine $(2,c)$ and $(2,d)$ in the following profile.

a	a	a	b
c	b	c	c
d	$\boldsymbol{c}$	d	$\widetilde{y}$
b	$\boldsymbol{d}$	b	$\widetilde{z}$

  – $(1,b)$, $(3,b)$, and $(4,a)$ are determined by efficiency, and consequently a is determined by ETA. $(2,d)$ is determined by considering the profile where agent 2 swaps $\{b,c\}$. That profile's disagreement parameter equals that of the original profile,[53] and agents 1 and 3 share the same ranking in it, so that profile is determined by Remark 19. $(4,b)$ is determined by considering the profile where agent 4 ranks $bacd$. That profile's disagreement parameter is less than that of the original profile.[54] Finally, $(2,c)$ is determined by complementarity.

a	$\boldsymbol{b}$	a	b
b	c	c	$\widetilde{x}$
c	a	x	$\widetilde{y}$
d	d	y	$\widetilde{z}$

- When dP_4c, we use this profile to determine $(2,b)$. $(2,a)$, $(3,b)$, and $(4,c)$ are determined by efficiency. $(1,c)$ is determined by considering the profile where agent 1 swaps $\{a,b\}$. If aP_4c, that profile is determined by the lemma; otherwise, its disagreement parameter equals that of the original profile, and it is not supported.[55] $(2,d)$ is determined by considering the profile where agent 2 ranks $abcd$ and applying the lemma. $(3,a)$ and $(3,d)$ are determined by considering the profile where agent 3 ranks $abcd$.[56] That profile's disagreement parameter is at most that of the original profile,[57] and agents 1 and 3 share the same ranking in it, so that profile is determined by Remark 19. Consequently, $(2,b)$ is determined by complementarity.

Case iii. When agent 3 ranks $adxy$, the constraints require that cP_4d. Thus, by Remark 29, the remaining possibilities for agent 4's ranking are of the form

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[52] The swap of $\{b,c\}$ in agent 3's ranking and of $\{a,c\}$ (and possibly $\{a,d\}$) in agent 4's ranking do not cause any of these pairs to lose their near-unanimous agreement.

[53] Swapping $\{b,c\}$ and $\{b,d\}$ in agent 1's ranking each increases the disagreement parameter by 1, whereas swapping $\{b,a\}$ and $\{b,c\}$ in agent 2's ranking each decreases it by 1.

[54] The swap of $\{c,a\}$ in agent 4's ranking decreases the disagreement parameter by 3, and if dP_4a, then the swap of $\{d,a\}$ decreases it even further.

[55] When cP_4a, the swap of $\{a,c\}$ in agent 2's ranking increases the parameter by only 1, while the swap $\{a,b\}$ in agent 1's ranking decreases it by 1. In the resulting profile, agents 2 and 4 are not supported because of $\{c,a\}$ and $\{d,c\}$, respectively, and agent 3 is not supported because of the pair containing b and the house immediately above it.

[56] We use the fact that agent 3 cannot receive b even when bP_3d, together with Remark 4 to justify that the same transition also determines $(3,d)$ when dP_3b.

[57] Swapping $\{c,b\}$ in agent 3's ranking decreases the disagreement parameter by 3, which is at least as large as the increase produced by swapping $\{a,c\}$ in agent 2's ranking. An additional swap of $\{d,b\}$ in agent 3's ranking, if it occurs, may reduce the parameter even further.

$bc\widetilde{x}\widetilde{y}$, since agent 2 ranks $bacd$.

$\underline{a}$	b	a	b
b	a	d	c
c	c	x	$\widetilde{x}$
d	d	y	$\widetilde{y}$

(23)

In profiles of this form, agent 3 is not supported because of the pair containing b and the house immediately above it, and similarly agent 4 when the same reasoning is applied with a in place of b. It then suffices to determine $(1,a)$ in the following profile.

a	b	a	b
c	a	d	c
b	c	x	$\widetilde{x}$
d	d	y	$\widetilde{y}$

$(1,b)$, $(3,b)$, and $(4,a)$ are determined by efficiency, and consequently b is determined by ETA. $(2,c)$ and $(2,d)$ are determined by considering the profile where agent 2 swaps $\{b,a\}$. If bP_3c, that profile is determined by the lemma; otherwise, its disagreement parameter equals that of the original profile and it is not supported.[58] $(3,a)$ is determined by considering the profile where agent 3 ranks $abcd$. That profile's disagreement parameter is less than that of the original profile.[59] $(1,a)$ is then determined by complementarity.

Case 7. When agent 2 ranks $badc$, agents 1 and 2 disagree on both adjacent pairs $\{a,b\}$ and $\{c,d\}$, and by Remark 20, agents 3 and 4 cannot satisfy the constraints.

Case 8. When agent 2 ranks $bcad$, by Remark 22, we must have $L_1 = (a,b)$, or $L_2 = (c,a)$, or $L_1 = L_2 = (b,c)$. Note that in the first two cases, we have $R_{cd} = R_{bc} = 1$, so by Remarks 24 and 26 we may assume that agent 3 ranks either a or b first.

Case i. When $L_1 = (a,b)$, by Remark 27 we may assume that agent 3 does not rank b first. Hence, agent 3 ranks a first, then agent 2 cannot receive a. Hence, we must have $L_2 \neq (c,a)$, but then $R_{ca} = 2$, which requires cP_3a, a contradiction.

Case ii. When $L_2 = (c,a)$, we may assume $L_1 = \varnothing$ or $L_1 = (c,d)$, since $L_1 = (b,c)$ is redundant by Remark 21, and the case $L_1 = (a,b)$ was already addressed. If agent 3 ranks a first, then agent 2 cannot receive a. Otherwise, agent 3 ranks b first, but this violates the constraint $R_{ab} = 2$.

Case iii. When $L_1 = L_2 = (b,c)$, the constraints $R_{ca} = R_{ab} = 2$ imply that both agents 3 and 4 rank c above a and a above b. Hence, agent 3 ranks c first, and agent 4 ranks either c or d first, but in either case agent 1 cannot receive c.

Case 9. When agent 2 ranks $bcda$:

Case i. When $L_1 = L_2 = \varnothing$, the constraints fix the rankings of agents 3 and 4 to

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[58] In this case, the swap of $\{b,c\}$ in agent 1's ranking increased the disagreement parameter by only 1, which is exactly offset by the decrease induced by the swap of $\{b,a\}$ in agent 2's ranking. In the resulting profile, agents 1 and 3 are not supported because of the pair $\{c,b\}$, and agent 4 is not supported because of the pair containing a and the house immediately above it.

[59] Here, the swaps of $\{d,b\}$ and $\{d,c\}$ in agent 3's ranking reduce the parameter by 3 each, and the increment caused by swapping $\{c,b\}$ in agent 1's ranking is less than the total reduction of these swaps.

be $cdab$ and $dabc$, respectively, which yields a degenerate profile.

Case ii. When $L_2 = (d, a)$, agent 2 cannot receive a.

Case iii. When $L_1 = (a, b)$, we have already handled the case $L_2 = (d, a)$, and the cases $L_2 = (b, c)$ and $L_2 = (c, d)$ are redundant by Remark 21, so we may assume $L_2 = \varnothing$. Moreover, since $R_{bc} = R_{cd} = 1$, by Remarks 24, 26, and 27, we may assume that agent 3 ranks a first, but this violates the constraint $R_{da} = 2$.

Case iv. When $L_1 = L_2 = (b, c)$, the constraints determine agent 3's ranking to be $cdab$, and agent 4 must rank d first, yielding a degenerate profile.

Case v. When $L_1 = L_2 = (c, d)$, the constraints require that either agent 3 or agent 4 rank d first, but then agent 1 cannot receive d.

Case 10. When agent 2 ranks $bdac$, note that $\widetilde{R}^2_{da} = \widetilde{R}^1_{ab} = 2$ and $\widetilde{R}^2_{bd} = 1$. By Remark 22, this implies that $L_1 = (a, b)$, or $L_2 = (d, a)$, or $L_2 = (b, d)$. Similarly, since $\widetilde{R}^1_{cd} = \widetilde{R}^2_{da} = 2$ and $\widetilde{R}^2_{ac} = 1$, Remark 22 also implies that $L_1 = (c, d)$, or $L_2 = (d, a)$, or $L_2 = (a, c)$.

Case i. When $L_1 = \varnothing$ or $L_1 = (b, c)$, the arguments above imply that we must have $L_2 = (d, a)$. However, the constraints $R_{ab} = R_{cd} = 2$ and $R_{ac} = 1$ imply that either agent 3 or agent 4 ranks a first. In that case, agent 2 cannot receive a.

Case ii. When $L_1 = (a, b)$, the arguments above imply that $L_2 = (d, a)$ or $L_2 = (a, c)$. Moreover, since $R_{bc} = 1$ and $R_{cd} = 2$, by Remarks 24, 26, and 27, we may assume that agent 3 ranks a first, but then agent 2 cannot receive a.

Case iii. When $L_1 = (c, d)$ and $L_2 = (b, d)$, the constraints fix the rankings of agents 3 and 4 to be $cdab$ and $dabc$, respectively. This yields a degenerate profile.

Case iv. Otherwise, $L_1 = (c, d)$ and $L_2 = (d, a)$. Since $R_{bc} = R_{cd} = 1$, by Remarks 24 and 26, we may assume that agent 3 ranks either a or b first. If agent 3 ranks a first, agent 2 cannot receive a, and if he ranks b first, this contradicts $R_{ab} = 2$.

Case 11. When agent 2 ranks $bdca$, Remark 20 implies that we must have $L_1 = (c, d)$ and $L_2 = (d, c)$. However, even after these relaxations, the constraints still cannot be satisfied, since $R_{ca} = R_{ab} = 2$ and $R_{bc} = 1$.

Case 12. When agent 2 ranks $caxy$, we may rename the houses so that his ranking becomes $abcd$. Under this renaming, agent 1's ranking becomes one that ranks b first. Swapping the names of agents 1 and 2 then reduces this case to a previously analyzed one.

Case 13. When agent 2 ranks $cbad$, agents 1 and 2 disagree on both adjacent pairs $\{a, b\}$ and $\{b, c\}$. By Remark 20, agents 3 and 4 cannot satisfy the constraints.

Case 14. When agent 2 ranks $cbda$, Remark 20 implies that we must have $L_1 = (b, c)$ and $L_2 = (c, b)$. However, even after these relaxations, the constraints still cannot be satisfied, since $R_{da} = R_{ab} = 2$ and $R_{bd} = 1$.

Case 15. When agent 2 ranks $cdab$:

Case i. When $L_1 = L_2 = \varnothing$, the constraints fix the rankings of agents 3 and 4 to be $bcda$ and $dabc$, respectively, which yields a degenerate profile.

Case ii. When $L_1 = \varnothing$ and $L_2 = (d, a)$, since $R_{cd} = 1$, by Remark 24, we may assume that agent 3 does not rank d first. If agent 3 ranks a first, then agent 2 cannot receive a. If agent 3 ranks b first, the constraints $R_{ab} = 1$ and $R_{bc} = 2$ imply that agent 4 must rank d first, and we obtain a degenerate profile. Otherwise, agent 3 ranks c first, but then the constraint $R_{bc} = 2$ is not satisfied.

Case iii. When $L_1 = L_2 = (a, b)$, the constraints $R_{bc} = 2$, $R_{cd} = 1$, and $R_{da} = 2$ force one of agents 3 or 4 to rank b first, but then agent 2 cannot receive b.

Case iv. When $L_1 = L_2 = (c, d)$, the constraints force one of agents 3 or 4 to rank d first, but then agent 1 cannot receive d.

Case v. When $L_1 = (b, c)$, we may assume that $L_2 = \varnothing$ or $L_2 = (d, a)$. Since $R_{cd} = 1$, by Remark 24, we may assume that agent 3 does not rank d first. If agent 3 ranks a first, agent 2 cannot receive a. Thus we may assume $L_2 \neq (d, a)$. But then $R_{da} = 2$ implies dP_3a, a contradiction. If agent 3 ranks b first, then the constraint $R_{ab} = 1$ implies aP_4b, so agent 4 must rank either c or d first. If he ranks c first, agent 1 cannot receive c; if he ranks d first, we obtain a degenerate profile. Otherwise, agent 3 ranks c first, in which case agent 1 cannot receive c.

Case 16. When agent 2 ranks $cdba$, Remark 20 requires the relaxation $L_2 = (b, a)$. However, this relaxation is admissible only when agent 2 can receive a, and in this case agent 2 cannot receive a.

Case 17. When agent 2 ranks d first and does not rank a last, we may rename the houses so that his ranking becomes $abcd$. Under this renaming, agent 1's ranking becomes one that ranks either b or c first. Swapping the names of agents 1 and 2 then reduces this case to a previously analyzed one.

Case 18. When agent 2 ranks $dbca$:

Case i. When $L_i = \varnothing$ for some $i \in \{1, 2\}$, the constraints force one of agents 3 or 4 to share the same ranking with agent i.

Case ii. When $L_1 = (a, b)$ and $L_2 = (d, b)$, since $R_{cd} = R_{ca} = 2$, we may assume that agent 3 does not rank either a or d first. If agent 3 ranks b first, $R_{ab} = 1$ and $R_{cd} = 2$ force agent 4 to rank c first, and we obtain a degenerate profile. Otherwise, agent 3 ranks c first, and then agent 4 must rank either c or d first, but this violates $R_{bc} = 1$ or $R_{cd} = 2$, respectively.

Case iii. When $L_1 = L_2 = (b, c)$, the constraints $R_{cd} = R_{ca} = R_{ab} = 2$ imply that agents 3 and 4 must rank c first. In that case, agents 1 and 2 cannot receive c.

Case iv. Otherwise, either $L_1 = (c, d)$ or $L_2 = (c, a)$. However, agent 1 cannot receive d and agent 2 cannot receive a, so these relaxations cannot be applied.

Case 19. Otherwise, agent 2 ranks $dcba$. In this case, agents 1 and 2 disagree on the adjacent pairs $\{a, b\}$, $\{b, c\}$, and $\{c, d\}$. By Remark 20, agents 3 and 4 cannot satisfy the constraints.

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